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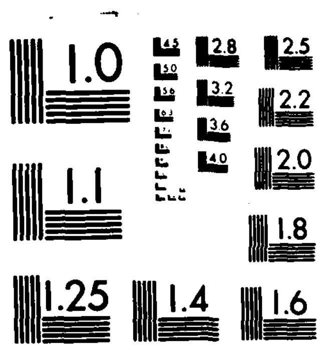


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# SMOOTH SOLUTIONS TO A QUASI-LINEAR SYSTEM OF DIFFUSION EQUATIONS FOR A CERTAIN POPULATION MODEL

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# SMOOTH SOLUTIONS TO A QUASI-LINEAR SYSTEM OF DIFFUSION EQUATIONS FOR A CERTAIN POPULATION MODEL

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## ABSTRACT

We prove the existence of smooth nonnegative solutions to the initial-boundary value problem associated with the system of diffusion equations which describes a certain population model:

$$(*)\quad \begin{cases} u_t = \Delta(c_1 u + d_1 uv) + (E_1 - a_1 u - b_1 v)u \\ v_t = \Delta(c_2 v + d_2 uv) + (E_2 - a_2 u - b_2 v)v, \quad (t,x) \in [0,\infty) \times [0,1] \end{cases}$$

$$(**)\qquad u(0,x) = u_0(x), \qquad v(0,x) = v_0(x)$$

$$(***)\qquad u_x(t,0) = u_x(t,1) = v_x(t,0) = v_x(t,1) = 0 ,$$

where $u$ and $v$ denote the densities of two competing species. Using Sobolevski's method, we establish the local existence of nonnegative solutions under the hypothesis $c_i > 0$, $d_i > 0$, $E_i > 0$, $a_i > 0$ and $b_i > 0$, $i = 1,2$. Under the additional hypothesis $c_1 = c_2$, we prove the global existence of solutions by energy estimates.

AMS (MOS) Subject Classifications: 35K55, 35K60, 35B65, 92A15

Key Words: system of diffusion equations, population model, smooth nonnegative solutions, Sobolevski's method, local solutions, energy estimates, global solutions

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## SIGNIFICANCE AND EXPLANATION

The system of diffusion equations (*) (see Abstract) proposed by Kawasaki, Shigesada and Teramoto describes a population model of two competing species with self- and cross-population pressures. The densities of the two species are denoted by $u$ and $v$. In this paper ~~we study~~ The author studies the initial-boundary value problem associated with (*). The Neumann boundary condition (***) corresponds to the case where the flux is zero at the boundary. Many investigators have considered nonlinear diffusion systems arising from various physical and biological problems. These equations, however, have a special structure: the highest order derivatives are not coupled or, at least, the coefficient matrix for the highest order derivatives is positive definite. This is not the case for the system (*) and hence, some of the techniques which are effective for those systems are no longer applicable to our case. Nevertheless, we can still use Sobolevski's method (see Reference [2]) to establish the local existence of solutions. Under the special assumption $c_1 = c_2$ in (*), we can also prove the global existence of solutions by energy estimates. The unusual structure of (*) seems to make it difficult to settle the question of asymptotic stability of solutions.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

SMOOTH SOLUTIONS TO A QUASI-LINEAR SYSTEM OF DIFFUSION EQUATIONS

FOR A CERTAIN POPULATION MODEL

Jong Uhn Kim

## 0. Introduction

This paper deals with the initial-boundary value problem for the system of equations:

$$(0\text{-}1)\qquad \begin{cases} \tilde{u}_\tau = \Delta(c_1\tilde{u} + d_1\tilde{u}\tilde{v}) + (\tilde{E}_1 - \tilde{a}_1\tilde{u} - \tilde{b}_1\tilde{v})\tilde{u} \\ \tilde{v}_\tau = \Delta(c_2\tilde{v} + d_2\tilde{u}\tilde{v}) + (\tilde{E}_2 - \tilde{a}_2\tilde{u} - \tilde{b}_2\tilde{v})\tilde{v}, \qquad (\tau,x) \in [0,\infty) \times [0,1] , \end{cases}$$

where $c_i$, $d_i$, $\tilde{E}_i$, $\tilde{a}_i$ and $\tilde{b}_i$, $i = 1,2$, are nonnegative constants. This system of equations describes a model of two competing species with self- and cross-population pressures. Here, $\tilde{u}$ and $\tilde{v}$ denote the population densities of the two competing species. For the derivation of Equations (0-1), the reader is referred to [3]. From the physical consideration, $\tilde{u}$ and $\tilde{v}$ should be nonnegative and (0-1) is subject to the Neumann boundary condition:

$$(0\text{-}2)\qquad \tilde{u}_x(\tau,x) = \tilde{v}_x(\tau,x) = 0 \quad \text{at} \quad x = 0,1 .$$

For the case when $c_1 > 0$, $c_2 > 0$, $d_1 > 0$ and $d_2 = 0$, the stationary problem associated with (0-1) was discussed in [4]. Also in its introduction, it was announced that Masuda and Mimura have proved the global existence of nonnegative solutions to (0-1) in the above case.

In this paper we shall prove the existence of smooth nonnegative solutions to (0-1) with suitably smooth initial data under the assumption that $c_i > 0$, $d_i > 0$, $i = 1,2$. In Section 1, we establish the local existence of solutions by the method due to Sobolevski, which is well presented in [2]. We employ the function spaces $\Phi_s$, $s > 0$ (see Section 1), which enable us to prove the $C^\infty$-regularity of solutions for $t > 0$. Some properties of $\Phi_s$ which are necessary in the development of our arguments are proved in the Appendix. In Section 2, we prove that the local solutions can be extended globally on $[0,\infty)$ under

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the additional hypothesis that $c_1 = c_2 > 0$, but without any restriction on the size of initial data.

We shall make some remarks on the structure of (0-1). First of all, we see that (0-1) and (0-2) reduce to

$$\begin{cases} u_t = \Delta(u + uv) + (E_1 - a_1u - b_1v)u \\ v_t = \Delta(\gamma v + uv) + (E_2 - a_2u - b_2v)v , \end{cases} \tag{0-3}$$

$$u_x(t,x) = v_x(t,x) = 0 \quad \text{at} \quad x = 0,1 , \tag{0-4}$$

through

$$\begin{cases} \tilde{u}(\tau,x) = \dfrac{c_1}{d_2} u(c_1\tau,x) \\ \tilde{v}(\tau,x) = \dfrac{c_1}{d_1} v(c_1\tau,x) \\ c_1\tau = t , \end{cases} \tag{0-5}$$

where $\gamma = \dfrac{c_2}{c_1} > 0$, $E_i > 0$, $a_i > 0$ and $b_i > 0$, $i = 1,2$. Throughout this paper we will consider (0-3), (0-4) instead of (0-1), (0-2). It is interesting to observe some unusual features possessed by (0-3). For simplicity, we shall consider

$$\begin{cases} u_t = \Delta(u + uv) \\ v_t = \Delta(\gamma v + uv) \end{cases} \tag{0-6}$$

and the associated nonlinear operator $S$:

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} -\Delta(u + uv) \\ -\Delta(\gamma v + uv) \end{pmatrix} . \tag{0-7}$$

Then, for smooth nonnegative functions $u$ and $v$ satisfying the Neumann boundary condition, $\langle S\binom{u}{v},\binom{u}{v}\rangle_{L^2\times L^2}$ is not nonnegative in general. In fact, it is strictly negative if we take $u = 100 + \gamma + 6\cos\pi x$, $v = 10 - \cos\pi x$, for example. Hence, we expect to have difficulty in obtaining energy estimates to establish global existence of solutions to (0-6). Now the linear operator associated with (0-7) is

$$(0\text{-}8) \qquad \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} -(1+f)\Delta u - g\Delta v \\ -f\Delta u - (\gamma + g)\Delta v \end{pmatrix}$$

where $f$ and $g$ are assumed to be given nonnegative functions. Then it is easy to see that the right-hand side of (0-8) is not a strongly elliptic system in general. This also suggests that the usual procedure to obtain energy estimates may not be effective. However, if $\gamma = 1$, i.e. $c_1 = c_2$, then we can make use of $u - v$ as an intermediary function to obtain necessary energy estimates. This is illustrated in Section 2. Finally we report that the question of asymptotic stability of solutions remains open. In view of the above remarks, it seems hopeless to get a uniform bound of the solution through energy estimates. In the mean time, the structure of $S$ discourages us from attempting to construct an invariant set.

__Acknowledgement__. I am very grateful to Professor M. Crandall for his invaluable advice and encouragement throughout this work. In particular, he pointed out some serious errors in Section 1 and significantly simplified the original lengthy estimate of the $L_0^2$-norm in Section 2.

## Section 1. Local Existence

As mentioned above, we shall use the method in [2]. Hence, we write Equations (0-3) in the form of an abstract evolution equation and verify all the conditions prescribed in the above reference. Let us define the linear operator $A_s(t,w)$ as follows:

$$(1\text{-}1)\qquad A_s(t,w)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} R_s u - (1 + fe^{R_s t})\Delta u - ge^{R_s t}\Delta v \\ R_s v - fe^{R_s t}\Delta u - (\gamma + ge^{R_s t})\Delta v \end{pmatrix},$$

where $R_s$ is a positive constant which will be determined later on and $w = \binom{g}{f}$. Writing $u^* = ue^{-R_s t}$, $v^* = ve^{-R_s t}$, (0-3) is equivalent to

$$(1\text{-}2)\qquad \frac{d}{dt}\begin{pmatrix} u^* \\ v^* \end{pmatrix} + A_s\left(t, \begin{pmatrix} u^* \\ v^* \end{pmatrix}\right)\begin{pmatrix} u^* \\ v^* \end{pmatrix} = F_s\left(t, \begin{pmatrix} u^* \\ v^* \end{pmatrix}\right),$$

where

$$(1\text{-}3)\qquad F_s\left(t, \begin{pmatrix} u^* \\ v^* \end{pmatrix}\right) = \begin{pmatrix} 2e^{R_s t}u_x^* v_x^* + (E_1 - a_1 e^{R_s t}u^* - b_1 e^{R_s t}v^*)u^* \\ 2e^{R_s t}u_x^* v_x^* + (E_2 - a_2 e^{R_s t}u^* - b_2 e^{R_s t}v^*)v^* \end{pmatrix}.$$

From now on, we shall suppress "*" and use both notations $\binom{a}{b}$ and $(a,b)$ to denote the same vector. For real $s$, we define

$$(1\text{-}4)\qquad \Phi_s = \{\sum_{n=0}^{\infty} a_n \cos n\pi x : a_n \in \mathbb{C} \text{ and } \sum_{n=0}^{\infty} |a_n|^2(1 + n^2\pi^2)^s < \infty\},$$

and if $f = \sum_{n=0}^{\infty} a_n \cos n\pi x$ and $g = \sum_{n=0}^{\infty} b_n \cos n\pi x$ lie in $\Phi_s$, we write

$$(1\text{-}5)\qquad \langle f,g\rangle_s = a_0\overline{b_0} + \frac{1}{2}\sum_{n=1}^{\infty}(1 + n^2\pi^2)^s a_n\overline{b_n}$$

and

$$(1\text{-}6)\qquad \|f\|_s = \langle f,f\rangle_s^{1/2}.$$

Obviously, $\Phi_{s_1} \subset \Phi_{s_2}$ and $\|\cdot\|_{s_2} \le \|\cdot\|_{s_1}$ if $s_1 > s_2$. We also define $X_s = \Phi_s \times \Phi_s$, $\|(g,f)\|_{X_s} = \|g\|_s + \|f\|_s$, for all $(g,f) \in X_s$, and

$$(1\text{-}7) \qquad L_m^2 = \{f \in L^2[0,1] : \left(\frac{d}{dx}\right)^k f \in L^2[0,1], \ k = 1,\dots,m\} .$$

$\|\cdot\|_{L_m^2}$ is defined in an obvious way. When $X$ and $Y$ are Banach spaces, we denote by $B(X,Y)$ the set of all bounded linear operators from $X$ into $Y$. Let $f(x), g(x)$ be real-valued functions in $\Phi_{s+1}$, $s \geq 0$, such that

$$\|f\|_{s+1}, \|g\|_{s+1} < M < \infty$$

and

$$f(x), g(x) > \max\left(-\frac{1}{4}, -\frac{\gamma}{4}\right), \quad \text{for all} \quad x \in [0,1] .$$

Denote $A_s(0,(g,f))$ by $A_s$. Then we have:

Proposition 1.1. There is a number $R(s,M) > 1$ depending on $s,M$ such that if $R_s > R(s,M)$, $(\lambda I - A_s)^{-1}$ is a bounded linear operator on $X_s$ for all $\lambda \in \mathbf{C}$ with $\mathrm{Re}\lambda \leq 0$, and

$$(1\text{-}8) \qquad \|(\lambda I - A_s)^{-1}\|_{B(X_s,X_s)} \leq \frac{C(s,M)}{|\lambda| + 1}$$

holds where $C(s,M)$ is a positive constant which depends only on $s,M$ and is independent of $\lambda, R_s$. Furthermore, $(\lambda I - A_s)^{-1}$ is a bounded linear operator from $X_s$ into $X_{s+2}$ with

$$(1\text{-}9) \qquad \|(\lambda I - A_s)^{-1}\|_{B(X_s,X_{s+2})} \leq C(s,M), \quad \text{for all} \quad \lambda \in \mathbf{C},\ \mathrm{Re}\lambda \leq 0 .$$

(Proof). First we prove the above assertion in the case where $s = m$ is a nonnegative integer with $f,g \in \Phi_\sigma$, where $\sigma = \begin{cases} 1 & \text{if} \quad m = 0 \\ m & \text{if} \quad m \geq 1 \end{cases}$. Assume $\|f\|_\sigma, \|g\|_\sigma < M$ and $f(x), g(x) > \max\left(-\frac{1}{4}, -\frac{\gamma}{4}\right)$ *for all* $x \in [0,1]$. *Now we will follow the well-known procedure.* Suppose $\xi = \sum_{n=0}^{\infty} \xi_n \cos n\pi x \in \Phi_m$, $\eta = \sum_{n=0}^{\infty} \eta_n \cos n\pi x \in \Phi_m$, $u = \sum_{n=0}^{\infty} u_n \cos n\pi x \in \Phi_{m+2}$ and $v = \sum_{n=0}^{\infty} v_n \cos n\pi x \in \Phi_{m+2}$ satisfy the equations:

$$(1\text{-}10) \qquad \begin{cases} (\lambda - R_m)u + (1 + f_0)\Delta u + g_0 \Delta v = \xi \\ (\lambda - R_m)v + f_0 \Delta u + (\gamma + g_0)\Delta v = \eta , \end{cases}$$

where $\lambda$ is a complex number with $\mathrm{Re}\lambda \leq 0$, $R_m > 1$, and $f_0, g_0$ are constants such that $f_0, g_0 > \max\left(-\frac{1}{4}, -\frac{\gamma}{4}\right)$. Then, it is easily seen that for all $n \geq 0$,

$$(1\text{-}11)\qquad u_n = \frac{\{\lambda - R_m - (\gamma + g_0)n^2\pi^2\}\xi_n + g_0 n^2\pi^2\eta_n}{(\lambda - R_m)^2 - (\lambda - R_m)(1 + \gamma + f_0 + g_0)n^2\pi^2 + (\gamma + g_0 + \gamma f_0)n^4\pi^4}$$

and

$$(1\text{-}12)\qquad v_n = \frac{f_0 n^2\pi^2\xi_n + \{\lambda - R_m - (1 + f_0)n^2\pi^2\}\eta_n}{(\lambda - R_m)^2 - (\lambda - R_m)(1 + \gamma + f_0 + g_0)n^2\pi^2 + (\gamma + g_0 + \gamma f_0)n^4\pi^4}.$$

Now we will estimate $|u_n|$ and $|v_n|$. Setting $\lambda = -\mu + i\nu$, $\mu > 0$, $\nu \in R$, we can rewrite (1-11), (1-12):

$$(1\text{-}11)^* \quad u_n = \frac{(-\mu-R_m+i\nu)\xi_n-(\gamma+g_0)n^2\pi^2\xi_n+g_0n^2\pi^2\eta_n}{(\mu+R_m)^2+(\mu+R_m)(1+\gamma+f_0+g_0)n^2\pi^2+(\gamma+g_0+\gamma f_0)n^4\pi^4-\nu^2-i\nu\{2(\mu+R_m)+(1+\gamma+f_0+g_0)n^2\pi^2\}},$$

$$(1\text{-}12)^* \quad v_n = \frac{f_0n^2\pi^2\xi_n+(-\mu-R_m+i\nu)\eta_n-(1+f_0)n^2\pi^2\eta_n}{(\mu+R_m)^2+(\mu+R_m)(1+\gamma+f_0+g_0)n^2\pi^2+(\gamma+g_0+\gamma f_0)n^4\pi^4-\nu^2-i\nu\{2(\mu+R_m)+(1+\gamma+f_0+g_0)n^2\pi^2\}}.$$

Case 1. $(\mu+R_m) \le |\nu| \le |\nu|^2 \le 2\{(\mu+R_m)^2+(\mu+R_m)(1+\gamma+f_0+g_0)n^2\pi^2+(\gamma+g_0+\gamma f_0)n^4\pi^4\}$.

In this case, we use the inequality:

$$\{(\mu + R_m) + (1 + \gamma + f_0 + g_0)n^2\pi^2\}^2 \ge (\mu + R_m)^2 + (\mu + R_m)(1 + \gamma + f_0 + g_0)n^2\pi^2 + (\gamma + g_0 + \gamma f_0)n^4\pi^4$$

to derive

$$(1\text{-}13)\qquad |u_n| \le \frac{C}{|\lambda| + R_m}(|\xi_n| + |\eta_n|)$$

amd

$$(1\text{-}14)\qquad |v_n| \le \frac{C}{|\lambda| + R_m}(|\xi_n| + |\eta_n|)\ ,$$

where $C$ is a positive constant independent of $\lambda, f_0, g_0, \xi_n, \eta_n, n$ and $R_m$.

Case 2. $(\mu+R_m) \le |\nu| \le |\nu|^2$ and $|\nu|^2 \ge 2\{(\mu+R_m)^2+(\mu+R_m)(1+\gamma+f_0+g_0)n^2\pi^2+(\gamma+g_0+\gamma f_0)n^4\pi^4\}$.

Case 3. $\frac{1}{2}(\mu + R_m) \le |\nu| \le \mu + R_m$.

Case 4. $|\nu| \le \frac{1}{2}(\mu + R_m)$.

For Cases 2, 3, 4, it is easy to obtain (1-13), (1-14). Therefore, we conclude that (1-13), (1-14) hold for all $\lambda \in C$, $\mathrm{Re}\lambda \le 0$ and all $R_m \ge 1$ where $C$ is a positive constant independent of $\lambda, f_0, g_0, \xi_n, \eta_n, n$ and $R_m$. Next we shall estimate $\pi^2 n^2 |u_n|$ and $\pi^2 n^2 |v_n|$, for $n \neq 0$. From (1-11)*, (1-12)*, we get

$$\pi^2 n^2 u_n = \frac{(-x+iy)\xi_n - (\gamma+g_0)\xi_n + g_0\eta_n}{x^2 + (1+\gamma+f_0+g_0)x + (\gamma+g_0+\gamma f_0) - y^2 - iy(2x+1+\gamma+f_0+g_0)} \tag{1-15}$$

and

$$\pi^2 n^2 v_n = \frac{f_0\xi_n + (-x+iy)\eta_n - (1+f_0)\eta_n}{x^2 + (1+\gamma+f_0+g_0)x + (\gamma+g_0+\gamma f_0) - y^2 - iy(2x+1+\gamma+f_0+g_0)} \tag{1-16}$$

where $x = \dfrac{\mu + R_m}{n^2\pi^2} > 0, \quad y = \dfrac{\nu}{n^2\pi^2}$.

<u>Case 1</u>. $x^2 + \frac{1}{2}(\gamma + g_0 + \gamma f_0) < y^2 < 2\{x^2 + (1 + \gamma + f_0 + g_0)x + \gamma + g_0 + \gamma f_0\}$.

In this case, we use the inequality: $(x + 1 + \gamma + f_0 + g_0)^2 > x^2 + (1 + \gamma + f_0 + g_0)x + \gamma + g_0 + \gamma f_0$ to derive that

$$\pi^2 n^2 |u_n| < C(|\xi_n| + |\eta_n|) \tag{1-17}$$

and

$$\pi^2 n^2 |v_n| < C(|\xi_n| + |\eta_n|) , \tag{1-18}$$

where $C$ is a positive constant independent of $\lambda$, $f_0$, $g_0$, $\xi_n$, $\eta_n$, $n$ and $R_m$.

<u>Case 2</u>. $2\{x^2 + (1 + \gamma + f_0 + g_0)x + \gamma + g_0 + \gamma f_0\} < y^2$.

<u>Case 3</u>. $y^2 < x^2 + \frac{1}{2}(\gamma + g_0 + \gamma f_0)$.

In Cases 2, 3, it is easy to get (1-17), (1-18). Therefore, we conclude that

$$\|u\|_m + \|v\|_m < \frac{C}{|\lambda| + R_m}(\|\xi\|_m + \|\eta\|_m) \tag{1-19}$$

and

$$\|u\|_{m+2} + \|v\|_{m+2} < C(\|\xi\|_m + \|\eta\|_m) \tag{1-20}$$

hold for all $\lambda \in \mathbb{C}$, all $R_m > 1$, where $C$ is a positive constant independent of $\lambda$, $f_0$, $g_0$, $\xi$, $\eta$ and $R_m$. Next suppose $u \in \Phi_{m+2}$, $v \in \Phi_{m+2}$, $\xi \in \Phi_m$ and $\eta \in \Phi_m$ satisfy the following equations:

$$\begin{cases} (\lambda - R_m)u + (1 + f)\Delta u + g\Delta v = \xi \\ (\lambda - R_m)v + f\Delta u + (\gamma + g)\Delta v = \eta \end{cases} \tag{1-21}$$

where $\lambda \in \mathbf{C}$, $\mathrm{Re}\lambda \le 0$, $R_m \ge 1$, and $f,g \in \Phi_\sigma$ are real-valued functions satisfying $\|f\|_\sigma, \|g\|_\sigma \le M$ and $f(x), g(x) \ge \max\left(-\frac{1}{4}, -\frac{\gamma}{4}\right)$ for all $x \in [0,1]$. Let us choose a partition of unity $\{\phi_1,\ldots,\phi_N\}$ as follows:

(i) $\phi_i \ge 0$, $\phi_i \in C^\infty([0,1])$, $\sum_{i=1}^{N} \phi_i \equiv 1$ on $[0,1]$ and $\left(\frac{d}{dx}\right)^k \phi_i(x) = 0$ at $x = 0,1$, for all $k \ge 1$;

(ii) $\mathrm{supp}\ \phi_i \subset [0,1] \cap [x_i - d_i, x_i + d_i]$, $x_i \in [0,1]$, for some $d_i > 0$ and $|f(x_i) - f(x)| \le \varepsilon$, $|g(x_i) - g(x)| \le \varepsilon$ hold for all $x \in [0,1] \cap [x_i - 2d_i, x_i + 2d_i]$.

Note that the choice of $N$, $\{\phi_1,\ldots,\phi_N\}$ depends on $\varepsilon$ and $M$. Next we define a set of functions $\{\psi_1,\ldots,\psi_N\}$ such that for each $i = 1,\ldots,N$,

(i)* $0 \le \psi_i \le 1$, $\psi_i \in C_0^\infty(-\infty,\infty)$;

(ii)* $\psi_i \equiv 1$ on $[x_i - d_i, x_i + d_i]$, $\psi_i \equiv 0$ on $(-\infty,\infty) - [x_i - 2d_i, x_i + 2d_i]$.

$\varepsilon > 0$ will be determined later on. By multiplying (1-21) by $\phi_i$, we see that

$$(1\text{-}22)\qquad \begin{aligned}(\lambda - R_m)(u\phi_i) &+ (1 + f_i)\Delta(u\phi_i) + g_i\Delta(v\phi_i) \\ &= \xi\phi_i + (1 + f)u\Delta\phi_i + 2(1 + f)u_x\phi_{ix} + gv\Delta\phi_i + 2gv_x\phi_{ix} \\ &\quad + (f_i - f)\Delta(u\phi_i) + (g_i - g)\Delta(v\phi_i)\end{aligned}$$

and

$$(1\text{-}23)\qquad \begin{aligned}(\lambda - R_m)(v\phi_i) &+ f_i\Delta(v\phi_i) + (\gamma + g_i)\Delta(v\phi_i) = \\ &= \eta\phi_i + fu\Delta\phi_i + 2fu_x\phi_{ix} + (\gamma + g)v\Delta\phi_i + 2(\gamma + g)v_x\phi_{ix} \\ &\quad + (f_i - f)\Delta(u\phi_i) + (g_i - g)\Delta(v\phi_i)\ ,\end{aligned}$$

where $f_i = f(x_i)$, $g_i = g(x_i)$, $i = 1,\ldots,N$. From the Appendix, it is easy to see that the right-hand sides of (1-22) and (1-23) lie in $\Phi_m$ with the following estimates:

$$(1\text{-}24)\qquad \|(f_i - f)\Delta(u\phi_i)\|_0 \le \varepsilon\|\Delta(u\phi_i)\|_0\ ;$$

$$(1\text{-}24)^*\qquad \begin{aligned}\|(f_i - f)\Delta(u\phi_i)\|_1 &\le C\|\psi_i(f_i - f)\|_{L^\infty}\|\Delta(u\phi_i)\|_1 + C\|\psi_i(f_i - f)\|_1\|\Delta(u\phi_i)\|_{\frac{3}{4}} \\ &\le \varepsilon C\|\Delta(u\phi_i)\|_1 + C\|\psi_i(f_i - f)\|_1\|\Delta(u\phi_i)\|_1^{\frac{3}{4}}\|\Delta(u\phi_i)\|_0^{\frac{1}{4}} \\ &\le 2\varepsilon C\|\Delta(u\phi_i)\|_1 + \frac{C}{\varepsilon^3}\|\psi_i(f_i - f)\|_1^4\|\Delta(u\phi_i)\|_0\ ,\end{aligned}$$

where $C$ denotes positive constants independent of $\varepsilon$, $u$, $\phi_i$, $\psi_i$ and $f$;

$$(1\text{-}24)^{**}\qquad \|(f_i - f)\Delta(u\phi_i)\|_m \le C_m(\|\psi_i(f_i - f)\|_{L^\infty}\|\Delta(u\phi_i)\|_m + \|\psi_i(f_i - f)\|_{L^2_m}\|\Delta(u\phi_i)\|_{m-1})\ ,$$

for $m \geq 2$, where $C_m$ a positive constant which depends only on $m$;

(1-25) $$\|\xi\phi_i\|_m \leq C_m\|\xi\|_m\|\phi_i\|_\sigma, \quad \text{for all } m \geq 0 ;$$

(1-26) $$\|(1 + f)u\Delta\phi_i\|_m \leq C_m\|1 + f\|_\sigma\|u\|_m\|\Delta\phi_i\|_\sigma, \quad \text{for all } m \geq 0 ;$$

(1-27) $$\|(1 + f)u_x\phi_{ix}\|_m \leq C_m\|1 + f\|_\sigma\|u\|_{m+1}\|\phi_i\|_{\sigma+1}, \quad \text{for all } m \geq 0 .$$

The estimates for the remaining terms on the right-hand sides of (1-22), (1-23) are similar to (1-24) to (1-27). Thus, (1-20) yields

(1-28) $$\begin{aligned}\|u\phi_i\|_{m+2} + \|v\phi_i\|_{m+2} \leq C_m\{&\varepsilon\|\Delta(u\phi_i)\|_m + \varepsilon\|\Delta(v\phi_i)\|_m + \\ &+ (1 + \frac{1}{\varepsilon^3})(\|\psi_i(f_i - f)\|^4_{L^2_m} + \|\psi_i(g_i - g)\|^4_{L^2_m} + \|\psi_i(f_i - f)\|_{L^2_m} + \\ &+ \|\psi_i(g_i - g)\|_{L^2_m})(\|\Delta(u\phi_i)\|_{m-1} + \|\Delta(v\phi_i)\|_{m-1}) + \\ &+ (\|\xi\|_m + \|\eta\|_m)\|\phi_i\|_\sigma + (1 + \gamma + \|f\|_\sigma + \|g\|_\sigma)(\|u\|_{m+1} \\ &+ \|v\|_{m+1})(\|\Delta\phi_i\|_\sigma + \|\phi_i\|_{\sigma+1})\} ,\end{aligned}$$

for all $m \geq 0$, it being understood that $\|\cdot\|_{m-1} = 0$ if $m = 0$. $C_m$ is a positive constant which depends only on $m$ and is independent of $\varepsilon$, $\lambda$, $R_m$, $u$, $v$, $\phi_i$, $\psi_i$, $\xi$, $\eta$, $f$ and $g$. Hence, we could have taken $\varepsilon$ so small that $\varepsilon C_m < \frac{1}{2}$ at the outset. This, in turn, determines $N$, $\phi_1,\ldots,\phi_N$ and $\psi_1,\ldots,\psi_N$, depending only on $\varepsilon$, $M$. Now we suppose that such $\varepsilon$ was fixed and that the corresponding set of functions $\phi_1,\ldots,\phi_N$, $\psi_1,\ldots,\psi_N$ were chosen for given $M$. Then from (1-28), we find that

(1-29) $$\|u\phi_i\|_{m+2} + \|v\phi_i\|_{m+2} \leq C(m,M)(\|\xi\|_m + \|\eta\|_m + \|u\|_{m+1} + \|v\|_{m+1}) ,$$

where $C(m,M)$ denotes a positive constant depending only on $m$ and $M$. By summing (1-29) over $i = 1,\ldots,N$, we deduce that

(1-30) $$\|u\|_{m+2} + \|v\|_{m+2} \leq C(m,M)(\|\xi\|_m + \|\eta\|_m + \|u\|_{m+1} + \|v\|_{m+1}) ,$$

which, combined with the inequality

(1-31) $$\|u\|_{m+1} \leq C_m\|u\|^{\frac{1}{2}}_{m+2}\|u\|^{\frac{1}{2}}_m ,$$

gives

(1-32) $$\|u\|_{m+2} + \|v\|_{m+2} \leq C(m,M)(\|\xi\|_m + \|\eta\|_m + \|u\|_m + \|v\|_m) ,$$

where $C(m,M)$ denotes positive constants depending only on $m,M$. Now (1-21) is equivalent to

(1-33) $$\begin{cases}(\lambda - R_m)u + \Delta u = \xi - f\Delta u - g\Delta v \\ (\lambda - R_m)v + \gamma\Delta v = \eta - f\Delta u - g\Delta v .\end{cases}$$

Combined with (1-32), (1-19), applied to (1-33), yields

$$(1\text{-}34)\qquad \|u\|_m + \|v\|_m \le \frac{C}{|\lambda| + R_m}(\|\xi\|_m + \|\eta\|_m + 2\|f\Delta u\|_m + 2\|g\Delta v\|_m)$$

$$\le \frac{C(m,M)}{|\lambda| + R_m}(\|\xi\|_m + \|\eta\|_m + \|u\|_m + \|v\|_m)\ ,$$

for all $\lambda \in \mathbf{C}$, $\mathrm{Re}\lambda \le 0$ and all $R_m \ge 1$. Here, $C(m,M)$ is independent of $\lambda$ and $R_m$, and we may take $C(m,M) \ge \frac{1}{2}$. With this particular $C(m,M)$, we define:

$$(1\text{-}35)\qquad R(m,M) = 2C(m,M)\ .$$

So for all $R_m \ge R(m,M)$ and all $\lambda \in \mathbf{C}$, $\mathrm{Re}\lambda \le 0$, we have

$$(1\text{-}36)\qquad \|u\|_m + \|v\|_m \le \frac{C(m,M)}{|\lambda| + 1}(\|\xi\|_m + \|\eta\|_m)\ ,$$

which, together with (1-32), implies

$$(1\text{-}37)\qquad \|u\|_{m+2} + \|v\|_{m+2} \le C(m,M)(\|\xi\|_m + \|\eta\|_m)\ ,$$

where $C(m,M)$ is independent of $\lambda$ and $R_m$. Now the proof of the case $s = m$ is completed by the following lemma:

<u>Lemma 1.2</u>. Suppose $f,g$ are real-valued functions in $\Phi_\sigma$, $\|f\|_\sigma$, $\|g\|_\sigma \le M$ and $f(x),g(x) \ge \max\left(-\frac{1}{4},-\frac{\gamma}{4}\right)$ for all $x \in [0,1]$. Let $\lambda \in \mathbf{C}$, $\mathrm{Re}\lambda \le 0$ and $R_m \ge R(m,M)$. Then, for each $\xi,\eta \in \Phi_m$, there exist unique $u,v \in \Phi_{m+2}$ such that (1-21) holds.

(Proof). We will use the method of continuity. Consider the following equations with parameter $\mu$:

$$(1\text{-}38)\qquad \begin{cases} (\lambda - R_m)u + (1 + \mu f)\Delta u + \mu g\Delta v = \xi \\ (\lambda - R_m)v + \mu f\Delta u + (\gamma + \mu g)\Delta v = \eta\ . \end{cases}$$

Let us define $S = \{\mu \in [0,1]:$ for each $\xi,\eta, \in \Phi_m$, there exist unique $u,v \in \Phi_{m+2}$ such that (1-38) holds$\}$.

It is obvious that $0 \in S$. Suppose $\mu_0 \in S$ and consider the mapping $T_{\xi,\eta,\mu}$ from $X_{m+2}$ into $X_{m+2}$ defined by

$$(1\text{-}39)\qquad (\tilde{u},\tilde{v}) \longmapsto (u,v)\ ,$$

where $(u,v)$ is the unique solution of

$$
(1\text{-}40)\quad \begin{cases} (\lambda - R_m)u + (1 + \mu_0 f)\Delta u + \mu_0 g\Delta v = \xi + (\mu_0 - \mu)f\Delta\hat{u} + (\mu_0 - \mu)g\Delta\hat{v} \\ (\lambda - R_m)v + \mu_0 f\Delta u + (\gamma + \mu_0 g)\Delta v = \eta + (\mu_0 - \mu)f\Delta\hat{u} + (\mu_0 - \mu)g\Delta\hat{v} . \end{cases}
$$

With the aid of (1-37), we can choose $\delta > 0$ independent of $\xi, \eta$ such that $|\mu_0 - \mu| < \delta$ implies $T_{\xi,\eta,\mu}$ is a contraction for all $\xi, \eta$. The fixed point of $T_{\xi,\eta,\mu}$ is the unique solution of (1-38). Hence, $S$ is open. It is easy to see that $S$ is also closed. Therefore, $S = [0,1]$.

We proceed to consider the case where $s > 0$ is not an integer. Let $k \geq 1$ be an integer such that $k - 1 < s < k$. Suppose $f,g$ are real-valued functions in $\phi_k$, $\|f\|_k, \|g\|_k \leq M$, and $f(x), g(x) \geq \max\left(-\frac{1}{4}, -\frac{\gamma}{4}\right)$ for all $x \in [0,1]$. Then, we can determine $R(k,M)$ and $R(k-1,M)$ by (1-35). Let

$$
(1\text{-}41)\qquad R(s,M) = \max(R(k,M), R(k-1,M))
$$

and $R_s$ be any positive number such that $R_s \geq R(s,M)$. By taking $R_{k-1} = R_k = R_s$, we define $A_s \overset{\text{def}}{=} A_{k-1}(0,(g,f)) = A_k(0,(g,f))$. Then, we have proved that for all $\lambda \in \mathbf{C}$, $\mathrm{Re}\lambda \leq 0$, and for all $R_s \geq R(s,M)$,

$$
(1\text{-}42)\qquad (\lambda I - A_s)^{-1} \in B(X_{k-1}, X_{k-1}) \cap B(X_{k-1}, X_{k+1}) \cap B(X_k, X_k) \cap B(X_k, X_{k+2}) .
$$

By interpolation, we can conclude that

$$
(1\text{-}43)\qquad (\lambda I - A_s)^{-1} \in B(X_s, X_s) \cap B(X_s, X_{s+2})
$$

and that

$$
(1\text{-}44)\qquad \|u\|_s + \|v\|_s \leq \frac{C(s,M)}{|\lambda| + 1}\left(\|\xi\|_s + \|\eta\|_s\right) ,
$$

$$
(1\text{-}45)\qquad \|u\|_{s+2} + \|v\|_{s+2} \leq C(s,M)\left(\|\xi\|_s + \|\eta\|_s\right) ,
$$

where $C(s,M)$ denotes positive constants which depend only on $s,M$. Now the proof of Proposition 1.1 is complete.

Next we shall discuss some properties concerning $A_s(t,(g,f))$ and $F_s(t,(g,f))$.

<u>Lemma 1.3</u>. Let $(g_i, f_i) \in X_{s+1}$, $i = 1,2,3$, $s \geq 0$, such that $\|g_i\|_{s+1}, \|f_i\|_{s+1} \leq \frac{M}{2}$ and $g_i(x), f_i(x) \geq \frac{1}{2}\max\left(-\frac{1}{4}, -\frac{\gamma}{4}\right)$ for all $x \in [0,1]$, $i = 1,2,3$. Let $R_s \geq R(s,M)$ which is determined by (1-35) if $s$ is an integer and by (1-41) if $s$ is not an integer. Using this $R_s$, we define $A_s(t,(g_i,f_i))$. Let $T_s$ be a positive number such that $e^{R_s T_s} \leq 2$. Then for all $t_i \in [0,T_s]$, $i = 1,2,3$, it holds that

(1-46) $$\|\{A_s(t_1,(g_1,f_1)) - A_s(t_2,(g_2,f_2))\}A_s(t_3,(g_3,f_3))^{-1}\|_{B(X_s,X_s)}$$

$$\leq C(s,M,R_s)(|t_1 - t_2| + \|g_1 - g_2\|_{s+1} + \|f_1 - f_2\|_{s+1}) ,$$

where $C(s,M,R_s)$ is a positive constant which depends only on $s,M$ and $R_s$.

(Proof). First of all, by (1-1) and Proposition 1.1, we observe that $A_s(t_3,(g_3,f_3))^{-1} \in B(X_s,X_s) \cap B(X_s,X_{s+2})$. Set $(u,v) = A_s(t_3,(g_3,f_3))^{-1}(\xi,\eta)$. Then,

(1-47) $$\begin{cases} R_s u - (1 + f_3 e^{R_s t_3})\Delta u - g_3 e^{R_s t_3}\Delta v = \xi \\ R_s v - f_3 e^{R_s t_3}\Delta u - (\gamma + g_3 e^{R_s t_3})\Delta v = \eta \end{cases}$$

and

(1-48) $$\{A_s(t_1,(g_1,f_1)) - A_s(t_2,(g_2,f_2))\}(u,v) = \begin{pmatrix} (f_2 e^{R_s t_2} - f_1 e^{R_s t_1})\Delta u + (g_2 e^{R_s t_2} - g_1 e^{R_s t_1})\Delta v \\ (f_2 e^{R_s t_2} - f_1 e^{R_s t_1})\Delta u + (g_2 e^{R_s t_2} - g_1 e^{R_s t_1})\Delta v \end{pmatrix}.$$

Using the inequalities in the Appendix, it is easy to see that for all $t_1,t_2 \in [0,T_s]$,

(1-49) $$\begin{aligned} &\|\{A_s(t_1,(g_1,f_1)) - A_s(t_2,(g_2,f_2))\}(u,v)\|_{X_s} \leq \\ &\leq C(s,M,R_s)(|t_1 - t_2| + \|f_1 - f_2\|_{s+1} + \|g_1 - g_2\|_{s+1})(\|u\|_{s+2} + \|v\|_{s+2}) \\ &\leq C(s,M,R_s)(|t_1 - t_2| + \|f_1 - f_2\|_{s+1} + \|g_1 - g_2\|_{s+1})(\|\xi\|_s + \|\eta\|_s) \end{aligned}$$

from (1-37) and (1-45), where $C(s,M,R_s)$ denotes positive constants depending only on $s$, $M$ and $R_s$.

Next let us define

(1-50) $$\rho = \begin{cases} \frac{1}{4} + \frac{3}{4}s & \text{if } 0 \leq s < 1 \\ s & \text{if } s \geq 1 \end{cases}$$

and $F_s(t,(g,f))$ as in (1-3) using any $R_s$.

<u>Lemma 1.4</u>. Suppose $(g_i,f_i) \in X_{\rho+1}$ with $\|g_i\|_{\rho+1},\|f_i\|_{\rho+1} \leq M$, $i = 1,2$. Let $T_s$ be a number such that $e^{T_s R_s} \leq 2$. Then, for all $t_1,t_2 \in [0,T_s]$, it holds that

(1-51) $$\begin{aligned} &\|F_s(t_1,(g_1,f_1)) - F_s(t_2,(g_2,f_2))\|_{X_s} \leq \\ &\leq C(s,M,R_s)(|t_1 - t_2| + \|g_1 - g_2\|_{\rho+1} + \|f_1 - f_2\|_{\rho+1}) , \end{aligned}$$

where $C(s,M,R_s)$ is a positive constant depending only on $s$, $M$ and $R_s$.

(Proof). We write

(1-52) $F_s(t_1,(g_1,f_1)) - F_s(t_2,(g_2,f_2)) =$

$$= \begin{pmatrix} 2(e^{R_s t_1} g_{1x} f_{1x} - e^{R_s t_2} g_{2x} f_{2x}) + (E_1 - a_1 e^{R_s t_1} g_1 - b_1 e^{R_s t_1} f_1) g_1 - (E_1 - a_1 e^{R_s t_2} g_2 \\ - b_1 e^{R_s t_2} f_2) g_2 \\ 2(e^{R_s t_1} g_{1x} f_{1x} - e^{R_s t_2} g_{2x} f_{2x}) + (E_2 - a_2 e^{R_s t_1} g_1 - b_2 e^{R_s t_1} f_1) f_1 - (E_2 - a_2 e^{R_s t_2} g_2 \\ - b_2 e^{R_s t_2} f_2) f_2 \end{pmatrix} .$$

With the aid of the inequalities in the Appendix, we can estimate the right-hand side of (1-52):

$$\text{(1-53)} \qquad \| e^{R_s t_1} g_{1x} f_{1x} - e^{R_s t_2} g_{2x} f_{2x} \|_s \le | e^{R_s t_1} - e^{R_s t_2} | \, \| g_{1x} f_{1x} \|_s$$

$$+ e^{R_s t_2} \| g_{1x} f_{1x} - g_{2x} f_{2x} \|_s$$

$$\le C(s,M,R_s)(|t_1 - t_2| + \| g_1 - g_2 \|_{\rho+1} + \| f_1 - f_2 \|_{\rho+1}) ,$$

$$\text{(1-54)} \qquad \| e^{R_s t_1} f_1 g_1 - e^{R_s t_2} f_2 g_2 \|_s \le C(s,M,R_s)(|t_1 - t_2| + \| f_1 - f_2 \|_\rho + \| g_1 - g_2 \|_\rho) .$$

The remaining estimates are similar to (1-54) and the proof is complete.

Lemma 1.5. Let $f,g$ satisfy the same conditions as in Proposition 1.1. Define $A_s = A_s(0,(g,f))$ with $R_s > R(s,M)$. Then $\mathcal{D}(A_s^\mu)$ is continuously imbedded into $X_{\rho+1}$ if $\mu > \theta$ where $\theta = \frac{1}{2}$ if $s > 1$ and $\theta = \frac{5}{8} - \frac{s}{8}$ if $0 \le s \le 1$, and $X_{s+2\delta}$ is continuously imbedded into $\mathcal{D}(A_s^\mu)$ if $\delta > \mu$ and $0 < \mu < 1$. ($\mathcal{D}(A_s^\mu)$ is equipped with the graph norm.)

(Proof). First we note that $A_s$ is a linear operator in $X_s$ with $\mathcal{D}(A_s) = X_{s+2}$ and that the norm $\| \cdot \|_{X_{s+2}}$ is equivalent to the norm $\| A_s(\cdot) \|_{X_s}$. Therefore, it follows that

$$\text{(1-55)} \qquad \| x \|_{X_{\rho+1}} \le C(s,M) \| A_s x \|_{X_s}^{\theta} \| x \|_{X_s}^{1-\theta}, \quad \text{for all } x \in X_{s+2} ,$$

where $\theta = \frac{1}{2}$ if $s > 1$ and $\theta = \frac{5}{8} - \frac{s}{8}$ if $0 \le s \le 1$, and $C(s,m)$ depends only on $s$

and $M$. Combined with Lemma 17.1 of [2], (1-55) implies that $\mathcal{D}(\Lambda_s^{\mu})$ is continuously imbedded into $X_{\rho+1}$ if $\mu > \theta$. For the remaining assertion, we define the operator $Q = \begin{pmatrix} I - \Delta, & 0 \\ 0, & I - \Delta \end{pmatrix}$. Then, $Q$ is a positive-definite self-adjoint operator in $X_s$ with $\mathcal{D}(Q) = X_{s+2}$. Then, for all $x \in \mathcal{D}(Q)$, it holds that

$$\|x\|_{\mathcal{D}(\Lambda_s^{\mu})} \leq C(\mu,s,R_s,M)\|\Lambda_s x\|_{X_s}^{\mu}\|x\|_{X_s}^{1-\mu} \leq C(\mu,s,R_s,M)\|Qx\|_{X_s}^{\mu}\|x\|_{X_s}^{1-\mu} \tag{1-56}$$

where $C(\mu,s,R_s,M)$ denotes positive constants depending only on $\mu$, $s$, $R_s$ and $M$. Again using Lemma 17.1 of [2], we conclude that $\mathcal{D}(Q^{\delta})$ is continuously imbedded into $\mathcal{D}(\Lambda_s^{\mu})$ where $\delta > \mu$. Hence, $X_{s+2\delta}$ is continuously imbedded into $\mathcal{D}(\Lambda_s^{\mu})$ if $\delta > \mu$ and $0 < \mu < 1$.

Now we are ready to establish the local existence of solutions:

Proposition 1.6. Suppose $u_0(x)$, $v_0(x)$ are real-valued functions in $\Phi_{s+\nu}$, $\nu > \frac{5}{4}$, $s > 0$ such that $\|u_0\|_{s+\nu}, \|v_0\|_{s+\nu} < \frac{M}{4}$ and $u_0(x), v_0(x) > \frac{1}{4}\max\left(-\frac{1}{4}, -\frac{\gamma}{4}\right)$, for all $x \in [0,1]$. Let $R_s = R(s,M)$ and using this $R_s$, define $\Lambda_s = A_s(0,(u_0,v_0))$. Then, $\mathcal{D}(\Lambda_s) = X_{s+2}$ and there exists $t_s > 0$ such that (0-3) has a unique solution in $C^{\eta}([0,t_s]; \mathcal{D}(\Lambda_s^{\alpha})) \cap C((0,t_s]; \mathcal{D}(\Lambda_s))$ satisfying the initial condition $u(0,x) = u_0(x)$, $v(0,x) = v_0(x)$, where $\alpha$, $\beta$ and $\eta$ are positive numbers such that $\min\left(\frac{3}{4}, \frac{\nu}{2}\right) > \beta > \alpha > \frac{5}{8}$ and $0 < \eta < \beta - \alpha$. Here $t_s$ depends only on $\|(u_0,v_0)\|_{X_{s+\nu}}$, $\alpha$, $\beta$, $\eta$ and $s$.

(Recall that $R(s,M)$ is defined by (1-35) if $s$ is an integer and by (1-41) if $s$ is not an integer.)

(Proof). Choose $\alpha$, $\beta$ and $\eta$ such that $\min\left(\frac{3}{4}, \frac{\nu}{2}\right) > \beta > \alpha > \frac{5}{8}$ and $0 < \eta < \beta - \alpha$. Let $K$ be any positive number and define

$$Q_s(t_s,K,\eta) = \left\{ \begin{array}{l} y \in C^{\eta}([0,t_s]; X_s) : y(t) \text{ is real-vector} \\ \text{valued, } y(0) = \Lambda_s^{\alpha}(u_0,v_0) \text{ and} \\ \|y(t) - y(\tau)\|_{X_s} \leq K|t-\tau|^{\eta}, \ \forall t,\tau \in [0,t_s] \end{array} \right\}. \tag{1-57}$$

We take $t_s$ so small that

(1-58) $e^{R_s t_s} < 2$ ;

(1-59) $Kt_s^{\eta} < \frac{M}{4L(\alpha,s,M)}$, $L(\alpha,s,M)$ being the positive constant in the inequality

$$\max(\|g\|_{\rho+1}, \|h\|_{\rho+1}) \leq L(\alpha,s,M)\|\Lambda_s^{\alpha}(g,h)\|_{X_s}, \quad \text{for all } (g,h) \in \mathcal{D}(\Lambda_s^{\alpha}) ;$$

(1-60) $\quad Kt_s^{\eta} < -\frac{1}{4C(\alpha,s,M)}\max\left(-\frac{1}{4},-\frac{\gamma}{4}\right)$, $C(\alpha,s,M)$ being the positive constant in the inequality

$$\max(\|g\|_{L^\infty},\|h\|_{L^\infty}) \le C(\alpha,s,M)\|\Lambda_s^{\alpha}(g,h)\|_{X_s}, \quad \text{for all } (g,h) \in \mathcal{D}(\Lambda_s^{\alpha}) .$$

By virtue of (1-57) and (1-60), we find that for all $t \in [0,t_s]$, for all $y(t) \in Q_s(t_s,K,\eta)$,

(1-61)
$$\min\{p(t,x),q(t,x)\} > \frac{1}{2}\max\left\{-\frac{1}{4},-\frac{\gamma}{4}\right\}$$

holds for all $x \in [0,1]$, where $(p(t,x),q(t,x)) = \Lambda_s^{-\alpha}y(t)$. We write (0-3) as

(1-62)
$$\frac{d}{dt}z(t) + A_s(t,z(t))z(t) = F_s(t,z(t)) ,$$

where $z(t) = (e^{-R_s t}u(t,x),e^{-R_s t}v(t,x))$ and $R_s = R(s,M)$ as above. Let us define the mapping $L$ on $Q(t_s,K,\eta)$ as follows:

(1-63)
$$w(t) \longmapsto \Lambda_s^{\alpha}z_w(t) ,$$

where $z_w(t)$ is the unique solution of

(1-64)
$$\begin{cases} \frac{d}{dt}z(t) + A_s(t,\Lambda_s^{-\alpha}w(t))z(t) = F_s(t,\Lambda_s^{-\alpha}w(t)) \\ z(0) = (u_0,v_0) . \end{cases}$$

By virtue of (1-58), (1-59) and (1-61), it follows from Proposition 1.1 and Lemma 1.3 that for all $w \in Q_s(t_s,K,\eta)$ and all $t \in [0,t_s]$, $A_s(t,\Lambda_s^{-\alpha}w(t))$ is well-defined with $\mathcal{D}(A_s(t,\Lambda_s^{-\alpha}w(t))) = X_{s+2}$ and satisfies:

(1-65)
$$\|(\lambda I - A_s(t,\Lambda_s^{-\alpha}w(t)))^{-1}\|_{B(X_s,X_s)} \le \frac{C(s,M)}{|\lambda|+1} ,$$

for all $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda \le 0$, where $C(s,M)$ is independent of $t$, $w(t)$ and $\lambda$;

(1-66)
$$\begin{aligned}\|\{A_s(t_1,\Lambda_s^{-\alpha}w_1(t_1)) - A_s(t_2,\Lambda_s^{-\alpha}w_1(t_2))\}A_s(t_3,\Lambda_s^{-\alpha}w_2(t_3))^{-1}\|_{B(X_s,X_s)} \\ \le C(s,M)(|t_1-t_2| + KL(\alpha,s,M)|t_1-t_2|^{\eta}) ,\end{aligned}$$

for all $t_i \in [0,t_s]$, all $w_j \in Q(t_s,K,\eta)$, $i = 1,2,3$, $j = 1,2$. From Lemma 1.4, we see that for all $w \in Q(t_s,K,\eta)$ and all $t_i \in [0,t_s]$, $i = 1,2$,

(1-67)
$$\begin{aligned}\|F_s(t_1,\Lambda_s^{-\alpha}w(t_1)) - F_s(t_2,\Lambda_s^{-\alpha}w(t_2))\|_{X_s} \le \\ \le C(s,M)(|t_1-t_2| + KL(\alpha,s,M)|t_1-t_2|^{\eta}) .\end{aligned}$$

Since $\frac{\nu}{2} > \beta$, it is obvious that $(u_0,v_0) \in \mathcal{D}(\Lambda_s^{\beta})$ by Lemma 1.5. Now we can follow the procedure in [2]. Let us denote by $U_w(t,\tau)$ the fundamental solution corresponding to $A_s(t,\Lambda_s^{-\alpha}w(t))$ for $w \in Q_s(t_s,K,\eta)$. Then the solution $z_w(t)$ of (1-64) is given by

$$z_w(t) = U_w(t,0)(u_0,v_0) + \int_0^t U_w(t,\tau)F_s(\tau,\Lambda_s^{-\alpha}w(\tau))d\tau \tag{1-68}$$

and hence,

$$Lw(t) = \Lambda_s^{\alpha}U_w(t,0)(u_0,v_0) + \Lambda_s^{\alpha}\int_0^t U_w(t,\tau)F_s(\tau,\Lambda_s^{-\alpha}w(\tau))d\tau \ . \tag{1-69}$$

Using (1-65), (1-66) and (1-67), we can derive all the necessary estimates (see p. 172 - p. 174 of [2]) to conclude that $L$ maps $Q_s(t_s,K,\eta)$ into itself and has a unique fixed point $\bar{w}$ in $Q_s(t_s,K,\eta)$ by taking $t_s$ smaller if necessary. Hence, $\Lambda_s^{-\alpha}\bar{w}$ is a solution of (1-62) and the same calculation that shows $L$ is continuous yields the uniqueness of solution in the function class

$$C^{\eta}([0,t_s];\mathcal{D}(\Lambda_s^{\alpha})) \cap C((0,t_s];\mathcal{D}(\Lambda_s)) \ . \tag{1-70}$$

Next we shall show that the solution gains regularity for $t > 0$, starting from the case $s = 0$:

<u>Corollary 1.7</u>. Suppose $u_0(x)$, $v_0(x)$ are real-valued functions in $\Phi_\nu$, $\nu > \frac{5}{4}$ such that $\|u_0\|_\nu, \|v_0\|_\nu < \frac{M}{4}$ and $u_0(x), v_0(x) > 0$ for all $x \in [0,1]$. Using $R_0 = R(0,M)$, we define $\Lambda_0 = \Lambda_0(0,(u_0,v_0))$. Then there exists $t_0 > 0$ such that (0-3) has a unique solution in $C^{\eta}([0,t_0];\mathcal{D}(\Lambda_0^{\alpha})) \cap C((0,t_0];\mathcal{D}(\Lambda_0))$ satisfying the initial condition $u(0,x) = u_0(x)$, $v(0,x) = v_0(x)$, where $\alpha$ and $\eta$ are the numbers in the above proposition.

We take $t_0$ so small that

$$\|u(t,x)\|_{\frac{5}{4}}, \|v(t,x)\|_{\frac{5}{4}} < \frac{M}{2} \ ; \tag{1-71}$$

$$u(t,x), v(t,x) > \frac{1}{4}\max\left(-\frac{1}{4}, -\frac{\gamma}{4}\right), \quad \text{for all } t \in [0,t_0] \text{ and all } x \in [0,1] \ . \tag{1-72}$$

Now let $\xi_0(t) = (u(t,x),v(t,x))$ be the solution to (0-3) in the above corollary. Suppose $\tilde{\xi}_0(t) = (\tilde{u}(t,x),\tilde{v}(t,x))$ is a solution to (0-3) in $C^{\eta}([\delta,t_0];\mathcal{D}(\Lambda_0^{\alpha})) \cap C((\delta,t_0];\mathcal{D}(\Lambda_0))$, $0 < \delta < t_0$, satisfying $\tilde{\xi}_0(\delta) = \xi_0(\delta)$. Here $\Lambda_0$ is the same as in the corollary.

Writing $z(t) = e^{-R_0 t}\xi_0(t)$ and $\tilde{z}(t) = e^{-R_0 t}\tilde{\xi}_0(t)$, it is easily seen that both $z(t)$ and $\tilde{z}(t)$ are solutions of

$$(1\text{-}73)\qquad \begin{cases} \dfrac{d}{dt}\, y(t) + A_0(t,y(t))y(t) = F_0(t,y(t)) \\ \\ y(\delta) = e^{-R_0\delta}\xi_0(\delta) , \end{cases}$$

where $A_0(t,\cdot)$ and $F_0(t,\cdot)$ are defined with $R_0 = R(0,M)$ as in the above corollary. By virtue of (1-71), (1-72) and the fact that $\tilde{\xi}_0(t)$ lies in $C^{\eta}([\delta,t_0];\mathcal{D}(A_0^{\alpha})) \cap C([\delta,t_0];\mathcal{D}(A_0))$, we can use the argument in [2] to derive that $z(t) \equiv \tilde{z}(t)$ on $[\delta,\delta+\varepsilon]$ for some $\varepsilon > 0$. By repetition of the argument, we conclude that $z(t) \equiv \tilde{z}(t)$ on $[\delta,t_0]$. Next we observe that $A_0\xi_0(t)$ is uniformly Hölder continuous in $X_0$ on each compact subset of $(0,t_0]$. Fix any $t^* \in (0,t_0]$. Then, $\xi_0(\frac{t^*}{2}) \in X_2$. We take $\frac{t^*}{2}$ as the initial time and $\xi_0(\frac{t^*}{2})$ as the initial data, noting that $\xi_0(\frac{t^*}{2}) \in X_{\frac{1}{2}+\nu}$, $\nu = \frac{3}{2} > \frac{5}{4}$. In order to apply Proposition 1.6 to the case $s = \frac{1}{2}$, let us define $A_{\frac{1}{2}} = A_{\frac{1}{2}}(0,\xi_0(\frac{t^*}{2}))$ with $R_{\frac{1}{2}} = R(\frac{1}{2},M_{\frac{1}{2}})$ where $M_{\frac{1}{2}}$ is a positive number such that $\sup_{\tau\in[\frac{t^*}{2},t_0]} \|u(\tau,x)\|_2, \ \sup_{\tau\in[\frac{t^*}{2},t_0]} \|v(\tau,x)\|_2 < \frac{M_{\frac{1}{2}}}{4}$. From Proposition 1.6, there exists a number $\delta_1 > 0$ depending only on $M_{\frac{1}{2}}$ ($\alpha$, $\beta$ and $\eta$ are fixed) such that (0-3) has a unique solution $\xi_1(t)$ in $C^{\eta}([\frac{t^*}{2}, \frac{t^*}{2}+\delta_1];\mathcal{D}(A_{\frac{1}{2}}^{\alpha})) \cap C([\frac{t^*}{2}, \frac{t^*}{2}+\delta_1];\mathcal{D}(A_{\frac{1}{2}}))$ satisfying $\xi_2(\frac{t^*}{2}) = \xi_0(\frac{t^*}{2})$. Since $\mathcal{D}(A_{\frac{1}{2}}^{\alpha})$ is continuously imbedded into $\mathcal{D}(A_0^{\alpha})$, $\xi_1(t) \equiv \xi_0(t)$ on $[\frac{t^*}{2}, \frac{t^*}{2}+\delta_1] \cap [\frac{t^*}{2},t_0]$ by the uniqueness of solution. Now if we take any other point of $[\frac{t^*}{2},t^*]$ as our initial time and the corresponding $\xi_0(t)$ as our initial data, then $\delta_1$, the length of the time interval of existence, remains the same in view of the above argument. Therefore, if $\frac{t^*}{2} + \delta_1 < t_0$, we can extend $\xi_1(t)$ on the whole interval $[\frac{t^*}{2},t_0]$ such that $\xi_1(t) \in C([\frac{t^*}{2},t_0];X_{\frac{1}{2}+2})$ and $\xi_1(t) \equiv \xi_0(t)$ on $[\frac{t^*}{2},t_0]$. Consequently, $\xi_0(t) \in C([\frac{t^*}{2},t_0];X_{\frac{1}{2}+2})$. Next define $t_k^* = \frac{t^*}{2} + \cdots + \frac{t^*}{2^k}$ and suppose that

$\xi_0(t) \in C\big((t_k^*,t_0];X_{\frac{k}{2}+2}\big)$ has been proved. Then, $\xi_0(t_{k+1}^*) \in X_{\frac{k+1}{2}+\nu}$, $\nu = \frac{3}{2} > \frac{5}{4}$. Take $t_{k+1}^*$ as the initial time, $\xi_0(t_{k+1}^*)$ as the initial data, and define $A_{\frac{k+1}{2}} = A(0,\xi_0(t_{k+1}^*))$ with $R_{\frac{k+1}{2}} = R\big(\frac{k+1}{2},M_{\frac{k+1}{2}}\big)$, where $M_{\frac{k+1}{2}}$ is a positive number such that $\sup_{\tau\in[t_{k+1}^*,t_0]} \|u(\tau,x)\|_{\frac{k}{2}+2}$, $\sup_{\tau\in[t_{k+1}^*,t_0]} \|v(\tau,x)\|_{\frac{k}{2}+2} < \frac{1}{4} M_{\frac{k+1}{2}}$. Applying Proposition 1.6 to the case $s = \frac{k+1}{2}$, we obtain a local solution

$\xi_{k+1}(t) \in C^{\eta}\big([t_{k+1}^*,t_{k+1}^* + \delta_{k+1}];\mathcal{D}(A_{\frac{k+1}{2}}^{\alpha})\big) \cap C\big((t_{k+1}^*,t_{k+1}^* + \delta_{k+1}];\mathcal{D}(A_{\frac{k+1}{2}})\big)$, where $\delta_{k+1} > 0$ depends only on $M_{\frac{k+1}{2}}$. By the uniqueness of solution and the fact that $\mathcal{D}\big(A_{\frac{k+1}{2}}^{\alpha}\big)$ is continuously imbedded into $\mathcal{D}(A_0^{\alpha})$, $\xi_{k+1}(t) \equiv \xi_0(t)$ on $[t_{k+1}^*,t_{k+1}^* + \delta_{k+1}] \cap [t_{k+1}^*,t_0]$. As above, we can extend $\xi_{k+1}(t)$ on the whole interval $[t_{k+1}^*,t_0]$ to arrive at $\xi_0(t) \in C((t_{k+1}^*,t_0];X_{\frac{k+1}{2}+2})$. By induction, we conclude that $\xi_0(t) \in C([t^*,t_0];X_k)$ for all $k > 0$, and consequently, $u(t,x),v(t,x) \in C^{\infty}((0,t_0] \times [0,1])$ from (0-3) and the fact that $t^*$ was chosen arbitrarily.

Finally we shall prove that $u(t,x),v(t,x)$ are nonnegative. We may write (0-3) as

$$u_t = (1 + v)\Delta u + 2v_x u_x + \{\Delta v + (E_1 - a_1 u - b_1 v)\}u \,, \tag{1-74}$$

$$v_t = (\gamma + u)\Delta v + 2u_x v_x + \{\Delta u + (E_2 - a_2 u - b_2 v)\}v \,. \tag{1-75}$$

Since $\Delta v(t,x)$ and $\Delta u(t,x)$ may not be bounded near $t = 0$, we cannot apply the classical maximum principle directly to (1-74) or (1-75) to prove that $u(t,x),v(t,x) \geq 0$. However, the maximum principle can be used on the interval $[\delta,t_0]$ for any $\delta > 0$, since $u(t,x),v(t,x) \in C^{\infty}((0,t_0] \times [0,1])$. Thus, it is enough to prove that $u(t,x),v(t,x) \geq 0$ for all $x \in [0,1]$ and all $t \in [0,\delta]$, where $\delta$ is some positive number. For this purpose, let us denote by $(u_n(t,x),v_n(t,x))$ the solution to (0-3) with the initial condition $u_n(0,x) = u_0(x) + \frac{1}{n}$, $v_n(0,x) = v_0(x) + \frac{1}{n}$, $n \geq 1$. We choose $R_0 = R(0,M)$, where $M$ is the number such that $1 + \|u_0\|_{\nu}$, $1 + \|v_0\|_{\nu} < \frac{M}{4}$. Using this $R_0$, we define $A_0(t,\cdot)$, and write $\mathcal{A}_0 = A_0(0,(u_0,v_0))$, $\mathcal{A}_n = A_0(0,(u_0 + \frac{1}{n},v_0 + \frac{1}{n}))$. Now all the constants in the estimates to establish the local existence of solutions $(u(t,x),v(t,x))$, $(u_n(t,x),v_n(t,x))$, $n \geq 1$, can be taken uniformly with respect to $n$ (recall the proof of Proposition 1.6). Thus, there exists $\delta > 0$ independent of $n \geq 1$ such that $z(t) \in$

$C^n([0,\delta];\mathcal{D}(A_0^{\alpha})) \cap C((0,\delta];\mathcal{D}(A_0))$ is the solution of

$$
(1\text{-}76)\qquad \begin{cases} \dfrac{d}{dt} z(t) + A_0(t,z(t))z(t) = F_0(t,z(t)) \\ z(0) = (u_0,v_0) \end{cases}
$$

and $z_n(t) \in C^n([0,\delta];\mathcal{D}(A_n^{\alpha})) \cap C((0,\delta];\mathcal{D}(A_n))$ is the solution of

$$
(1\text{-}77)\qquad \begin{cases} \dfrac{d}{dt} z_n(t) + A_0(t,z_n(t))z_n(t) = F_0(t,z_n(t)) \\ z_n(0) = (u_0 + \dfrac{1}{n},\ v_0 + \dfrac{1}{n}) , \end{cases}
$$

where $z(t) = e^{-R_0 t}(u(t,x),v(t,x))$ and $z_n(t) = e^{-R_0 t}(u_n(t,x),v_n(t,x))$, $n \geq 1$. Choose any $\bar{\alpha}$ such that $\alpha > \bar{\alpha} > \frac{5}{8}$. Then, $\mathcal{D}(A_n^{\alpha})$ is continuously imbedded into $\mathcal{D}(A_0^{\bar{\alpha}})$ and consequently, $z(t),z_n(t) \in C^n([0,\delta];\mathcal{D}(A_0^{\bar{\alpha}})) \cap C((0,\delta];\mathcal{D}(A_0))$. Subtracting (1-77) from (1-76), we write

$$
(1\text{-}78)\qquad \frac{d}{dt}(z(t) - z_n(t)) + A_0(t,z(t))(z(t) - z_n(t)) =
$$
$$
= \{A_0(t,z_n(t)) - A_0(t,z(t))\}z_n(t) + F_0(t,z(t)) - F_0(t,z_n(t)) .
$$

Let $U(t,\tau)$ be the fundamental solution associated with $A_0(t,z(t))$. Following the argument in [2], we can write

$$
(1\text{-}79)\qquad z(t) - z_n(t) = U(t,\tau)(z(\tau) - z_n(\tau)) +
$$
$$
+ \int_\tau^t U(t,s)\{A_0(s,z_n(s)) - A_0(s,z(s))\}z_n(s)ds
$$
$$
+ \int_\tau^t U(t,s)\{F_0(s,z(s)) - F_0(s,z_n(s))\}ds
$$

for all $0 < \tau \leq t \leq \delta$, and subsequently, arrive at the inequality: for all $0 < \tilde{\delta} \leq \delta$,

$$
(1\text{-}80)\qquad \sup_{t\in[0,\tilde{\delta}]} \|z(t) - z_n(t)\|_{\mathcal{D}(A_0^{\bar{\alpha}})} \leq C\{\frac{1}{n} + \tilde{\delta}^{\beta-\bar{\alpha}} \sup_{t\in[0,\tilde{\delta}]} \|z(t) - z_n(t)\|_{\mathcal{D}(A_0^{\bar{\alpha}})}\} ,
$$

where $C$ is a positive constant independent of $n$ and $\tilde{\delta}$. Hence, for some $0 < \bar{\delta} \leq \delta$, $\|z_n(t) - z(t)\|_{\mathcal{D}(A_0^{\bar{\alpha}})} \to 0$ uniformly on $[0,\bar{\delta}]$, as $n \to \infty$, from which it follows that $u_n(t,x) \to u(t,x)$ and $v_n(t,x) \to v(t,x)$ uniformly on $[0,\bar{\delta}] \times [0,1]$. Since $u_n(t,x),v_n(t,x) \in C^{\infty}((0,\bar{\delta}] \times [0,1]) \cap C([0,\bar{\delta}] \times [0,1])$ and $u_n(0,x),v_n(0,x) \geq \frac{1}{n}$ for all $x \in [0,1]$, it is easily deduced that $u_n(t,x),v_n(t,x) \geq 0$ for all $(t,x) \in [0,\bar{\delta}] \times [0,1]$

with the aid of the maximum principle. Therefore we conclude that $u(t,x), v(t,x) > 0$ for all $(t,x) \in [0,\tilde{\delta}] \times [0,1]$. We have completed the proof of the main theorem:

<u>Theorem 1.8</u>. Suppose $u_0(x), v_0(x)$ are nonnegative, real-valued functions in $\Phi_\nu$, $\nu > \frac{5}{4}$. Then, there exists $t_0 > 0$ such that (0-3) has a unique solution in $C^\eta([0,t_0];\mathcal{D}(\Lambda_0^\alpha)) \cap [C^\infty((0,t_0] \times [0,1])]^2$ satisfying $u(0,x) = u_0(x)$, $v(0,x) = v_0(x)$ and $u_x(t,x) = v_x(t,x) = 0$ at $x = 0,1$, for all $t \in (0,t_0]$. Furthermore, $u(t,x), v(t,x) > 0$ for all $(t,x) \in [0,t_0] \times [0,1]$. ($\eta$, $\alpha$ and $\Lambda_0$ are the same as in Corollary 1.7.)

## Section 2. Global Existence of Solution

In the previous section, we obtained a unique solution $(u(t,x),v(t,x))$ to (0-3) satisfying $u(0,x) = u_0(x)$, $v(0,x) = v_0(x)$. Let $[0,T)$ be the maximal time interval to which $(u(t,x),v(t,x))$ can be extended so that $(u(t,x),v(t,x))$ lie in $C^n_{loc}([0,T);\mathcal{D}(\Lambda_0^\alpha)) \cap [C^\infty((0,T) \times [0,1])]^2$. Our purpose in this section is to prove that $T = \infty$ under the hypothesis $\gamma = 1$. In view of the local existence theorem, it is enough to prove that $\|u(t,x)\|_2, \|v(t,x)\|_2$ are bounded near $t = T$, assuming $T < \infty$. Assuming $\gamma = 1$, we write (0-3) as

$$u_t = \Delta(u + u^2 + u\zeta) + (E_1 - a_1u - b_1v)u \,, \tag{2-1}$$

$$v_t = \Delta(v + v^2 - v\zeta) + (E_2 - a_2u - b_2v)v \,, \tag{2-2}$$

$$\zeta_t = \Delta\zeta + G \,, \tag{2-3}$$

where $\zeta = v - u$ and $G = (E_2 - a_2u - b_2v)v - (E_1 - a_1u - b_1v)u$. The estimates will be obtained through three steps.

(Step 1) Multiplying (2-1), (2-2), (2-3) by $u, v, -\Delta\zeta$, respectively and integrating over $[0,1]$, we get, using the fact that $u(t,x), v(t,x) \geq 0$ and $u_x(t,0) = u_x(t,1) = v_x(t,0) = v_x(t,1) = 0$,

$$\frac{d}{dt}\frac{1}{2}\int_0^1 u^2dx \leq -\int_0^1 (1 + u)u_x^2dx + \frac{1}{2}\int_0^1 (\Delta\zeta)u^2dx + \int_0^1 E_1u^2dx \,, \tag{2-4}$$

$$\frac{d}{dt}\frac{1}{2}\int_0^1 v^2dx \leq -\int_0^1 (1 + v)v_x^2dx - \frac{1}{2}\int_0^1 (\Delta\zeta)v^2dx + \int_0^1 E_2v^2dx \,, \tag{2-5}$$

$$\frac{d}{dt}\frac{1}{2}\int_0^1 \zeta_x^2dx = -\int_0^1 (\Delta\zeta)^2dx - \int_0^1 (\Delta\zeta)Gdx \tag{2-6}$$

from which it follows that

$$\frac{d}{dt}\frac{1}{2}\int_0^1 (u^2 + v^2 + \zeta_x^2)dx \leq -\int_0^1 (1 + u)u_x^2dx - \int_0^1 (1 + v)v_x^2dx - \frac{1}{2}\int_0^1 (\Delta\zeta)^2dx \tag{2-7}$$

$$+ K\int_0^1 (u^2 + v^2)dx + \frac{1}{4}\int_0^1 (u^4 + v^4)dx, \quad \text{for all } t \in (0,T) \,,$$

where $K$ is a positive constant depending only on $E_1, E_2$. Integrating (2-1) and (2-2) over $[0,1]$, we obtain

$$\text{(2-8)} \qquad \frac{d}{dt}\int_0^1 u\,dx \le E_1 \int_0^1 u\,dx, \quad \text{for all } t \in (0,T),$$

and

$$\text{(2-9)} \qquad \frac{d}{dt}\int_0^1 v\,dx \le E_2 \int_0^1 v\,dx, \quad \text{for all } t \in (0,T)$$

from which follows

$$\text{(2-10)} \qquad \int_0^1 u\,dx + \int_0^1 v\,dx \le M_0, \quad \text{for all } t \in [0,T)$$

where $M_0$ is a positive constant depending only on the initial data, $E_1$, $E_2$ and $T$. From (2-10) and the inequality:

$$\text{(2-11)} \qquad \|f^2\|_{L^\infty} \le \left(1 + \frac{1}{\varepsilon}\right)\|f\|^2_{L^2} + \varepsilon\|f_x\|^2_{L^2}, \quad \text{for all } \varepsilon > 0, \text{ all } f \in L^2_1[0,1] ,$$

we find that

$$\text{(2-12)} \qquad \|u^3\|_{L^\infty} \le \left(1 + \frac{9}{4\varepsilon}\right)\int_0^1 u^3 dx + \varepsilon \int_0^1 uu_x^2 dx$$

$$\le \left(1 + \frac{9}{4\varepsilon}\right)M_0\|u^2\|_{L^\infty} + \varepsilon\int_0^1 uu_x^2 dx, \quad \text{for all } \varepsilon > 0, \text{ all } t \in [0,T) ,$$

and hence,

$$\text{(2-13)} \qquad -\int_0^1 uu_x^2 dx \le \frac{1}{\varepsilon}\left(1 + \frac{9}{4\varepsilon}\right)M_0\|u\|^2_{L^\infty} - \frac{1}{\varepsilon}\|u\|^3_{L^\infty}, \quad \text{for all } \varepsilon > 0, \text{ all } t \in [0,T) .$$

In the same way,

$$\text{(2-14)} \qquad -\int_0^1 vv_x^2 dx \le \frac{1}{\varepsilon}\left(1 + \frac{9}{4\varepsilon}\right)M_0\|v\|^2_{L^\infty} - \frac{1}{\varepsilon}\|v\|^3_{L^\infty}, \quad \text{for all } \varepsilon > 0, \text{ all } t \in [0,T) .$$

Substituting (2-13), (2-14) into (2-7) and using (2-10), we have

$$\text{(2-15)} \qquad \frac{d}{dt}\frac{1}{2}\int_0^1 (u^2 + v^2 + \zeta_x^2)dx \le -\int_0^1 (u_x^2 + v_x^2)dx - \frac{1}{2}\int_0^1 (\Delta\zeta)^2 dx$$

$$+ K\int_0^1 (u^2 + v^2)dx + \frac{1}{4}M_0(\|u\|^3_{L^\infty} + \|v\|^3_{L^\infty})$$

$$+ \frac{1}{\varepsilon}\left(1 + \frac{9}{4\varepsilon}\right)M_0\left(\|u\|^2_{L^\infty} + \|v\|^2_{L^\infty}\right) - \frac{1}{\varepsilon}\left(\|u\|^3_{L^\infty} + \|v\|^3_{L^\infty}\right),$$

which can be rewritten, after taking $\varepsilon = \frac{2}{M_0}$,

$$\text{(2-16)} \qquad \frac{d}{dt}\frac{1}{2}\int_0^1 (u^2 + v^2 + \zeta_x^2)dx \le -\int_0^1 (u_x^2 + v_x^2)dx - \frac{1}{2}\int_0^1 (\Delta\zeta)^2 dx$$

$$+ K\int_0^1 (u^2 + v^2)dx - \frac{1}{4}M_0(\|u\|^3_{L^\infty} + \|v\|^3_{L^\infty})$$

$$+ \frac{1}{2}M_0^2\left(1 + \frac{9}{8}M_0\right)(\|u\|^2_{L^\infty} + \|v\|^2_{L^\infty}), \quad \text{for all } t \in (0,T).$$

Since $-\frac{1}{4}M_0\tau^3 + \frac{1}{2}M_0^2(1 + \frac{9}{8}M_0)\tau^2 \le C(M_0)$ for all $\tau > 0$, we can apply Gronwall's inequality to deduce that

$$\text{(2-17)} \qquad \int_0^1 (u^2 + v^2 + \zeta_x^2)dx \le M_1, \quad \text{for all } t \in [0,T),$$

where $M_1$ is a positive constant depending on $E_1$, $E_2$, $T$, $\|u_0\|_{L^2}$, $\|v_0\|_{L^2}$ and $\|v_{0x} - u_{0x}\|_{L^2}$.

(Step 2) Multiplying (2-1), (2-2) by $-\Delta u, -\Delta v$, respectively, and integrating over $[0,1]$, we obtain

$$\text{(2-18)} \qquad \frac{1}{2}\frac{d}{dt}\int_0^1 u_x^2 dx \le -\int_0^1 (\Delta u)^2 dx - \int_0^1 u(\Delta u)^2 dx - \int_0^1 (u\Delta\zeta + 2u_x\zeta_x)\Delta u dx$$

$$+ K\int_0^1 u^2|\Delta u|dx + K\int_0^1 uv|\Delta u|dx + K\int_0^1 u_x^2 dx, \quad \text{for all } t \in (0,T)$$

and

$$\text{(2-19)} \qquad \frac{1}{2}\frac{d}{dt}\int_0^1 v_x^2 dx \le -\int_0^1 (\Delta v)^2 dx - \int_0^1 v(\Delta v)^2 dx + \int_0^1 (v\Delta\zeta + 2v_x\zeta_x)\Delta v dx$$

$$+ K\int_0^1 v^2|\Delta v|dx + K\int_0^1 uv|\Delta v|dx + K\int_0^1 v_x^2 dx, \quad \text{for all} \quad t \in (0,T)$$

where $K$ denotes positive constants depending only on $E_i, a_i, b_i$, $i = 1,2$. Applying the Laplacian $\Delta$ to both sides of (2-3), multiplying by $\Delta\xi$ and integrating over $[0,1]$, we have

$$\text{(2-20)} \qquad \frac{1}{2}\frac{d}{dt}\int_0^1 (\Delta\zeta)^2 dx \le -\frac{1}{2}\int_0^1 (\Delta\zeta_x)^2 dx + \frac{1}{2}\int_0^1 G_x^2 dx, \quad \text{for all} \quad t \in (0,T) .$$

Now we will estimate each term on the right-hand sides of (2-18), (2-19) and (2-20):

$$\text{(2-21)} \qquad \left|\int_0^1 u(\Delta\zeta)(\Delta u)dx\right| \le \|\Delta\zeta\|_{L^\infty}\|u\|_{L^2}\|\Delta u\|_{L^2} \le \frac{1}{8}\|\Delta u\|_{L^2}^2 + 2\|\Delta\zeta\|_{L^\infty}^2\|u\|_{L^2}^2$$

$$\le \frac{1}{8}\|\Delta u\|_{L^2}^2 + 2M_1\left\{(1+\frac{1}{\varepsilon})\|\Delta\zeta\|_{L^2}^2 + \varepsilon\|\Delta\zeta_x\|_{L^2}^2\right\}$$

$$\le \frac{1}{8}\|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\Delta\zeta_x\|_{L^2}^2 + C(M_1)\|\Delta\zeta\|_{L^2}^2, \quad \text{for all} \quad t \in (0,T) ,$$

which follows from (2-11) with $\varepsilon = \frac{1}{16M_1}$.

$$\text{(2-22)} \qquad \left|\int_0^1 2u_x\zeta_x\Delta u dx\right| = \left|\int_0^1 (\Delta\zeta)u_x^2 dx\right| \le \|\Delta\zeta\|_{L^\infty}\int_0^1 u_x^2 dx$$

$$= \|\Delta\zeta\|_{L^\infty}\int_0^1 (-uu_{xx})dx \le \|\Delta\zeta\|_{L^\infty}\|u\|_{L^2}\|\Delta u\|_{L^2}$$

$$\le \frac{1}{8}\|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\Delta\zeta_x\|_{L^2}^2 + C(M_1)\|\Delta\zeta\|_{L^2}^2, \quad \text{for all} \quad t \in (0,T) .$$

$$\text{(2-23)} \qquad K\int_0^1 uv|\Delta u|dx \le K^2\int_0^1 u^4 dx + K^2\int_0^1 v^4 dx + \frac{1}{8}\int_0^1 (\Delta u)^2 dx, \quad \text{for all} \quad t \in (0,T)$$

$$(2\text{-}24) \qquad \int_0^1 u^4 dx \le \|u\|^2_{L^\infty} \int_0^1 u^2 dx \le M_1\left(1 + \frac{1}{\varepsilon}\right)\|u\|^2_{L^2} + M_1 \varepsilon \|u_x\|^2_{L^2}, \quad \text{for all} \quad \varepsilon > 0$$

$$\text{and all} \quad t \in (0,T) .$$

Similar inequalities hold for $K \int_0^1 u^2 |\Delta u| dx$ and $\int_0^1 v^4 dx$. Combining these inequalities, we get

$$(2\text{-}25) \qquad K \int_0^1 u^2 |\Delta u| dx + K \int_0^1 uv |\Delta u| dx \le \frac{1}{4} \int_0^1 (\Delta u)^2 dx$$

$$+ C(M_1,K)(\|u_x\|^2_{L^2} + \|v_x\|^2_{L^2}) + C(M_1,K), \quad \text{for all} \quad t \in (0,T) .$$

The right-hand side of (2-19) can be estimated analogously to (2-21), (2-22) and (2-25).

$$(2\text{-}26) \qquad \int_0^1 G_x^2 dx \le K \int_0^1 (u_x^2 + v_x^2) dx + K \int_0^1 (u^2 + v^2)(u_x^2 + v_x^2) dx ,$$

$$\le K \int_0^1 (u_x^2 + v_x^2) dx + KM_1(\|u_x\|^2_{L^\infty} + \|v_x\|^2_{L^\infty})$$

$$\le K \int_0^1 (u_x^2 + v_x^2) dx + \frac{1}{4}(\|\Delta u\|^2_{L^2} + \|\Delta v\|^2_{L^2}) + C(M_1,K)(\|u_x\|^2_{L^2}$$

$$+ \|v_x\|^2_{L^2}), \quad \text{for all} \quad t \in (0,T) ,$$

where $K$ denotes positive constants depending only on $E_i, a_i, b_i$, $i = 1,2$. Now summing (2-18), (2-19) and (2-20), we find that

$$(2\text{-}27) \qquad \frac{d}{dt} \int_0^1 \{u_x^2 + v_x^2 + (\Delta\zeta)^2\} dx \le C(M_1,E_i,a_i,b_i) \int_0^1 \{u_x^2 + v_x^2 + (\Delta\zeta)^2\} dx$$

$$+ C(M_1,E_i,a_i,b_i), \quad \text{for all} \quad t \in (0,T) ,$$

from which we derive, using Gronwall's inequality, (2-18) $$\int_0^1 \{u_x^2 + v_x^2 + (\Delta\zeta)^2\}dx \le M_2, \quad \text{for all } t \in [\delta,T) ,$$

where $0 < \delta < T$ and $M_2$ is a positive constant depending on $\delta, T$ and $\|u_0(x)\|_\nu + \|v_0(x)\|_\nu$. Here we fix $\delta$ and proceed to the last step.

(Step 3) Apply the Laplacian $\Delta$ to both sides of (2-1), (2-2), (2-3), multiply the resulting equations by $\Delta u, \Delta v, -\Delta^2\zeta$, respectively, and integrate over $[0,1]$:

(2-29) $$\frac{1}{2}\frac{d}{dt}\int_0^1 (\Delta u)^2 dx \le -\int_0^1 (\Delta u_x)^2 dx - \int_0^1 u(\Delta u_x)^2 dx - \int_0^1 6u_x(\Delta u)(\Delta u_x)dx$$
$$-\int_0^1 3\zeta_x(\Delta u)(\Delta u_x)dx - \int_0^1 3u_x(\Delta\zeta)(\Delta u_x)dx$$
$$-\int_0^1 u(\Delta\zeta_x)(\Delta u_x)dx + \frac{1}{2}\int_0^1 (\Delta H)^2 dx + \frac{1}{2}\int_0^1 (\Delta u)^2 dx, \quad \text{for all } t \in (0,T) ,$$

where $H = (E_1 - a_1u - b_1v)u$,

(2-30) $$\frac{1}{2}\frac{d}{dt}\int_0^1 (\Delta v)^2 dx \le -\int_0^1 (\Delta v_x)^2 dx - \int_0^1 v(\Delta v_x)^2 dx - \int_0^1 6v_x(\Delta v)(\Delta v_x)dx$$
$$+\int_0^1 3\zeta_x(\Delta v)(\Delta v_x)dx + \int_0^1 3v_x(\Delta\zeta)(\Delta v_x)dx$$
$$+\int_0^1 v(\Delta\zeta_x)(\Delta v_x)dx + \frac{1}{2}\int_0^1 (\Delta J)^2 dx + \frac{1}{2}\int_0^1 (\Delta v)^2 dx, \quad \text{for all } t \in (0,T) ,$$

where $J = (E_2 - a_2u - b_2v)v$,

(2-31) $$\frac{1}{2}\frac{d}{dt}\int_0^1 (\Delta\zeta_x)^2 dx \le -\frac{1}{2}\int_0^1 (\Delta^2\zeta)^2 dx + \frac{1}{2}\int_0^1 (\Delta G)^2 dx .$$

As before, we will estimate each term on the right-hand sides of the above inequalities.

(A) Estimate of $\int_0^1 u_x(\Delta u)(\Delta u_x)dx$.

First we observe that

$$(2\text{-}32)\qquad \int_0^1 (\Delta u)^2 dx = -\int_0^1 u_x u_{xxx} dx \le \left(\int_0^1 u_x^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 u_{xxx}^2 dx\right)^{\frac{1}{2}} ,$$

$$(2\text{-}33)\qquad (\Delta u(t,x))^2 = 2\int_a^x u_{xx} u_{xxx} dx, \quad \text{for all} \quad x \in [0,1],\ t \in (0,T) ,$$

where $a$ is a point depending on $t$ such that $u_{xx}(t,a) = 0$, and hence,

$$(2\text{-}34)\qquad \|\Delta u\|_{L^\infty} \le \sqrt{2}\left(\int_0^1 u_{xx}^2 dx\right)^{\frac{1}{4}} \left(\int_0^1 u_{xxx}^2 dx\right)^{\frac{1}{4}} , \quad \text{for all} \quad t \in (0,T) .$$

Using (2-32), (2-33) and the inequality: $a^\lambda b^{1-\lambda} \le \lambda a + (1-\lambda)b$, for all $a,b > 0$, $0 < \lambda < 1$, we obtain

$$(2\text{-}35)\qquad \left|\int_0^1 u_x (\Delta u)(\Delta u_x) dx\right| = \frac{1}{2}\left|\int_0^1 (\Delta u)^3 dx\right| \le \frac{1}{2}\|\Delta u\|_{L^\infty} \int_0^1 (\Delta u)^2 dx$$

$$\le \frac{1}{\sqrt{2}}\left(\int_0^1 u_{xx}^2 dx\right)^{\frac{1}{4}}\left(\int_0^1 u_{xxx}^2 dx\right)^{\frac{3}{4}}\left(\int_0^1 u_x^2 dx\right)^{\frac{1}{2}}$$

$$\le \frac{1}{4}\,\frac{1}{4\varepsilon^3} M_2^2 \int_0^1 u_{xx}^2 dx + \frac{3\varepsilon}{4}\int_0^1 u_{xxx}^2 dx ,$$

for all $\varepsilon > 0$, all $t \in [\delta,T)$.

(B) Estimate of $\int_0^1 u_x(\Delta\zeta)(\Delta u_x) dx$ .

We write, by integration by parts,

$$\int_0^1 u_x(\Delta\zeta)(\Delta u_x) dx = -\int_0^1 (\Delta\zeta)(\Delta u)^2 dx - \int_0^1 u_x(\Delta\zeta_x)\Delta u dx .$$

Since $\|\Delta\zeta\|_{L^\infty} \le \left|\int_a^x \Delta\zeta_x dx\right| \le \left(\int_0^1 (\Delta\zeta_x)^2 dx\right)^{\frac{1}{2}}$, $a$ being the point at which $\Delta\zeta = 0$, we see that

$$(2\text{-}36)\qquad \left|\int_0^1 (\Delta\zeta)(\Delta u)^2 dx\right| \le \|\Delta\zeta\|_{L^\infty}\int_0^1 (-u_x u_{xxx})dx \le \left(\int_0^1 (\Delta\zeta_x)^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 u_x^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 u_{xxx}^2 dx\right)^{\frac{1}{2}}$$

$$\le M_2^{\frac{1}{2}}\left(\int_0^1 (\Delta\zeta_x)^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 u_{xxx}^2 dx\right)^{\frac{1}{2}},\quad \text{for all}\quad t \in [\delta,T)\ .$$

On the other hand,

$$(2\text{-}37)\qquad \left|\int_0^1 u_x(\Delta\zeta_x)\Delta u dx\right| = \frac{1}{2}\left|\int_0^1 (\Delta^2\zeta)(u_x^2)dx\right| \le \frac{1}{2}\|u_x^2\|_{L^\infty}\left(\int_0^1 (\Delta^2\zeta)^2 dx\right)^{\frac{1}{2}}$$

$$\le \left(\int_0^1 u_x^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 u_{xx}^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 (\Delta^2\zeta)^2 dx\right)^{\frac{1}{2}}$$

$$\le M_2^{\frac{1}{2}}\left(\int_0^1 u_{xx}^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 (\Delta^2\zeta)^2 dx\right)^{\frac{1}{2}},\quad \text{for all}\quad t \in [\delta,T)\ .$$

From (2-36), (2-37), we have

$$(2\text{-}38)\qquad \left|\int_0^1 u_x(\Delta\zeta)(\Delta u_x)dx\right| \le \varepsilon\int_0^1 u_{xxx}^2 dx + \varepsilon\int_0^1 (\Delta^2\zeta)^2 dx$$

$$+\frac{M_2}{4\varepsilon}\int_0^1 \{(\Delta\zeta_x)^2 + u_{xx}^2\}dx,\quad \text{for all}\quad \varepsilon > 0\ ,$$

all $t \in [\delta,T)$.

(C) Estimates of $\int_0^1 \zeta_x(\Delta u)(\Delta u_x)dx$ and $\int_0^1 u(\Delta\zeta_x)(\Delta u_x)dx$ .

It is easily seen that

$$(2\text{-}39)\qquad \|\zeta_x(t,x)\|_{L^\infty} \le \int_0^1 |\Delta\zeta| dx \le \left(\int_0^1 (\Delta\zeta)^2 dx\right)^{\frac{1}{2}} \le M_2^{\frac{1}{2}},\quad \text{for all}\quad t \in [\delta,T)\ ,$$

from which follows

$$(2\text{-}40)\qquad \left|\int_0^1 \zeta_x(\Delta u)(\Delta u_x)dx\right| \le \|\zeta_x\|_{L^\infty}\|\Delta u\|_{L^2}\|\Delta u_x\|_{L^2} \le M_2^{\frac{1}{2}}\|\Delta u\|_{L^2}\|\Delta u_x\|_{L^2}$$

$$\le \epsilon\|\Delta u_x\|_{L^2}^2 + \frac{M_2}{4\epsilon}\|\Delta u\|_{L^2}^2, \quad \text{for all} \quad \epsilon > 0 \quad \text{and all} \quad t \in [\delta,T).$$

With the aid of (2-11), we obtain

$$(2\text{-}41)\qquad \left|\int_0^1 u(\Delta\zeta_x)(\Delta u_x)dx\right| \le \|u\|_{L^\infty}\|\Delta\zeta_x\|_{L^2}\|\Delta u_x\|_{L^2} \le (M_2 + 2M_1)^{\frac{1}{2}}\|\Delta\zeta_x\|_{L^2}\|\Delta u_x\|_{L^2}$$

$$\le \epsilon\|\Delta u_x\|_{L^2}^2 + \frac{1}{4\epsilon}(M_2 + 2M_1)\|\Delta\zeta_x\|_{L^2}^2, \quad \text{for all} \quad \epsilon > 0$$

$$\text{and all} \quad t \in [\delta,T).$$

(D) Estimate of $\int_0^1 \{(\Delta H)^2 + (\Delta J)^2 + (\Delta G)^2\}dx$.

It is obvious that

$$\int_0^1 \{(\Delta H)^2 + (\Delta J)^2 + (\Delta G)^2\}dx \le K\int_0^1 (u^2 + v^2)\{(\Delta u)^2 + (\Delta v)^2\}dx +$$

$$+ K\int_0^1 (u_x^2 + v_x^2)^2 dx + K\int_0^1 \{(\Delta u)^2 + (\Delta v)^2\}dx ,$$

where $K$ denotes positive constants depending on $E_i, a_i, b_i$, $i = 1,2$. Using (2-11) with $\epsilon = 1$, we have

$$(2\text{-}42)\qquad \int_0^1 (u^2 + v^2)\{(\Delta u)^2 + (\Delta v)^2\}dx \le (\|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2)\int_0^1 \{(\Delta u)^2 + (\Delta v)^2\}dx$$

$$\le (M_2 + 2M_1)\int_0^1 \{(\Delta u)^2 + (\Delta v)^2\}dx , \quad \text{for all} \quad t \in [\delta,T).$$

Since

$$\|u_x\|^2_{L^\infty} + \|v_x\|^2_{L^\infty} \le \left(\int_0^1 |\Delta u|dx\right)^2 + \left(\int_0^1 |\Delta v|dx\right)^2 \le \int_0^1 (\Delta u)^2 dx + \int_0^1 (\Delta v)^2 dx, \quad \text{for all}$$

$t \in (0,T)$,

we find that

$$(2\text{-}43) \qquad \int_0^1 (u_x^2 + v_x^2)^2 dx \le M_2 \int_0^1 (\Delta u)^2 dx + M_2 \int_0^1 (\Delta v)^2 dx, \quad \text{for all} \quad t \in [\delta,T) .$$

The remaining estimates can be obtained similarly to (A), (B) and (C). Adding (2-29), (2-30), (2-31), and substituting the above inequalities into the right-hand side, we obtain by taking $\varepsilon$ sufficiently small

$$(2\text{-}44) \quad \frac{d}{dt}\int_0^1 \{(\Delta u)^2 + (\Delta v)^2 + (\Delta\zeta_x)^2\}dx \le C(E_i,a_i,b_i,M_1,M_2) \int_0^1 \{(\Delta u)^2 + (\Delta v)^2 + (\Delta\zeta_x)^2\}dx,$$
$$\text{for all} \quad t \in [\delta,T) ,$$

where $C(E_i,a_i,b_i,M_1,M_2)$ denotes a positive constant depending only on $E_i,a_i,b_i$, $i = 1,2$, and $M_1,M_2$. We deduce by Gronwall's inequality that

$$(2\text{-}45) \qquad \int_0^1 \{(\Delta u)^2 + (\Delta v)^2 + (\Delta\zeta_x)^2\}dx \le M_3, \quad \text{for all} \quad t \in [\delta,T) ,$$

where $M_3$ is a positive constant depending on $\delta$, $T$, $M_1$, $M_2$ and $\|u_0\|_\nu + \|v_0\|_\nu$. Combining the above estimates and Theorem 1.8, we can conclude:

<u>Theorem 2.1</u>. Suppose $\gamma = 1$ in (0-3) and $u_0(x),v_0(x)$ are real-valued, nonnegative functions in $\Phi_\nu$, $\nu > \frac{5}{4}$. Then, (0-3) has a unique global solution in $C^n_{loc}([0,\infty);\mathcal{D}(\Lambda_0^\alpha)) \cap [C^\infty((0,\infty) \times [0,1])]^2$ satisfying $u(0,x) = u_0(x)$, $v(0,x) = v_0(x)$ and $u_x(t,x) = v_x(t,x) = 0$ at $x = 0,1$, for all $t > 0$. Furthermore, it holds that $u(t,x),v(t,x) > 0$ for all $(t,x) \in [0,\infty) \times [0,1]$.

## Appendix

[A-1] Multiplication is a continuous bilinear map of $\Phi_{\frac{1}{2}-\varepsilon} \otimes \Phi_{\frac{1}{2}-\varepsilon}$ into $\Phi_0$ provided $\varepsilon < \frac{1}{4}$.

(Proof). Since $\Phi_0 = L_0^2$ and $\Phi_{\frac{1}{2}-\varepsilon}$ is continuously imbedded into $L^2_{\frac{1}{2}-\varepsilon}$ for any $\varepsilon < \frac{1}{2}$, the assertion follows from the fact that multiplication is a continuous bilinear map of $L^2_{\frac{1}{2}-\varepsilon} \otimes L^2_{\frac{1}{2}-\varepsilon}$ into $L_0^2$ for $\varepsilon < \frac{1}{4}$, which is a special case of Theorem 9.4 in [5].

[A-2] If $\varepsilon > 0$ and $m$ is a nonnegative integer, then multiplication is a continuous bilinear map of $\Phi_{\frac{1}{2}+\varepsilon+m} \otimes \Phi_m$ into $\Phi_m$.

(Proof). Let $f = \sum_{n=0}^{\infty} a_n \cos n\pi x \in \Phi_{\frac{1}{2}+\varepsilon+m}$ and $g = \sum_{n=0}^{\infty} b_n \cos n\pi x \in \Phi_m$. Define $f_k = \sum_{n=0}^{k} a_n \cos n\pi x$ and $g_k = \sum_{n=0}^{k} b_n \cos n\pi x$. Then, as $k \to \infty$, $f_k \to f$ in $\Phi_{\frac{1}{2}+\varepsilon+m}$, $g_k \to g$ in $\Phi_m$, and $f_k g_k \in \bigcap_{i=1}^{\infty} \Phi_i$ for each $k$. Now multiplication is a continuous bilinear mapping from $L^2_{\frac{1}{2}+\varepsilon+m} \otimes L_m^2$ into $L_m^2$ by Theorem 9.5 in [5]. Thus, $f_k g_k \to fg$ as $k \to \infty$ in $L_m^2$ since $\Phi_{\frac{1}{2}+\varepsilon+m}$ and $\Phi_m$ are continuously imbedded into $L^2_{\frac{1}{2}+\varepsilon+m}$ and $L_m^2$, respectively. The norm $\|\cdot\|_m$ is equivalent to the norm $\|\cdot\|_{L_m^2}$ and hence, $\{f_k g_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\Phi_m$, from which we deduce $fg \in \Phi_m$.

[A-3] Multiplication is a continuous bilinear mapping from $\Phi_{\frac{1}{2}+\varepsilon+s} \otimes \Phi_s$ into $\Phi_s$ provided $s > 0$, $\varepsilon > 0$.

(Proof). The assertion follows from [A-2] by interpolation [1].

[A-4] Multiplication is a continuous bilinear mapping from $\Phi_s \otimes \Phi_s$ into $\Phi_s$ provided $s > 1$.

(Proof). If $m > 1$ is an integer, the norm $\|\cdot\|_m$ is equivalent to the norm $\|\cdot\|_{L_m^2}$ and it is easy to see that multiplication is a continuous bilinear mapping from $L_m^2 \otimes L_m^2$ into $L_m^2$. Since $\Phi_m$ is continuously imbedded into $L_m^2$, the assertion follows when $s = m$, and the general case follows by interpolation.

[A-5] Multiplication is a continuous bilinear mapping from $\Phi_{\frac{1}{4}+\frac{3}{4}s} \otimes \Phi_{\frac{1}{4}+\frac{3}{4}s}$ into $\Phi_s$ provided $0 < s < 1$.

(Proof). By interpolation, the proof is immediate from [A-1] and [A-4].

[A-6] Define $\Gamma : (f,g) \longmapsto f_x g_x$. Then, $\Gamma$ is a continuous bilinear mapping:

(i) $\Phi_{\frac{5}{4}} \otimes \Phi_{\frac{5}{4}} \longrightarrow \Phi_0$ .

(ii) $\Phi_{m+1} \otimes \Phi_{m+1} \longrightarrow \Phi_m$, for all integers $m > 1$.

(Proof). (i) Suppose $f \in \Phi_{\frac{5}{4}}$ and $g \in \Phi_{\frac{5}{4}}$. Since $\Phi_{\frac{5}{4}}$ is continuously imbedded into $L^2_{\frac{5}{4}}$ and $\Phi_0 = L^2_0$, the assertion follows from Theorem 9.4 in [5].

(ii) Let $f = \sum_{n=0}^{\infty} a_n \cos n\pi x \in \Phi_{m+1}$ and $g = \sum_{n=0}^{\infty} b_n \cos n\pi x \in \Phi_{m+1}$. Define $f_k = \sum_{n=0}^{k} a_n \cos n\pi x$ and $g_k = \sum_{n=0}^{k} b_n \cos n\pi x$. Then, $f_{kx} g_{kx} \in \bigcap_{i=1}^{\infty} \Phi_i$ for each $k$, since $\sin(n\pi x)\sin(\ell\pi x) = \frac{1}{2}\{\cos(n-\ell)\pi x - \cos(n+\ell)\pi x\}$. In the mean time, $\Phi_{m+1}$ and $\Phi_m$ are continuously imbedded into $L^2_{m+1}$ and $L^2_m$, respectively, and the norms $\|\cdot\|_{m+1}, \|\cdot\|_m$ are equivalent to the norms $\|\cdot\|_{L^2_{m+1}}, \|\cdot\|_{L^2_m}$, respectively. Therefore, $f_{kx} g_{kx} \longrightarrow f_x g_x$ in $L^2_m$ and $\{f_{kx} g_{kx}\}_{k=1}^{\infty}$ is a Cauchy sequence in $\Phi_m$, from which the conclusion follows.

[A-7] $\Gamma$ is a continuous bilinear mapping:

(i) $\Phi_{\frac{5}{4}+\frac{3}{4}s} \otimes \Phi_{\frac{5}{4}+\frac{3}{4}s} \longrightarrow \Phi_s$ provided $0 < s < 1$ ;

(ii) $\Phi_{s+1} \otimes \Phi_{s+1} \longrightarrow \Phi_s$ provided $s > 1$ ;

(iii) $\Phi_{\frac{3}{2}+\varepsilon} \otimes \Phi_1 \longrightarrow \Phi_0$ provided $\varepsilon > 0$ ;

(iv) $\Phi_{\frac{5}{2}+\varepsilon} \otimes \Phi_2 \longrightarrow \Phi_1$ provided $\varepsilon > 0$ ;

(v) $\Phi_{\mu+1} \otimes \Phi_{s+1} \longrightarrow \Phi_s$ provided $s > 0$ and $s + \frac{1}{2} < \mu$ .

(Proof). (i) and (ii) follow from [A-6] by interpolation [1], and (iii) to (v) can be proved by the same argument as in [A-3], [A-6], and by interpolation.

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We prove the existence of smooth nonnegative solutions to the initial-boundary value problem associated with the system of diffusion equations which describes a certain population model:

$$(*)\quad \begin{cases} u_t = \Delta(c_1 u + d_1 uv) + (E_1 - a_1 u - b_1 v)u \\ v_t = \Delta(c_2 v + d_2 uv) + (E_2 - a_2 u - b_2 v)v, \quad (t,x) \in [0,\infty) \times [0,1] \end{cases}$$


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20. ABSTRACT - cont'd.

$$(**) \qquad u(0,x) = u_0(x),\ v(0,x) = v_0(x)$$

$$(***) \qquad u_x(t,0) = u_x(t,1) = v_x(t,0) = v_x(t,1) = 0\ ,$$

where $u$ and $v$ denote the densities of two competing species. Using Sobolevski's method, we establish the local existence of nonnegative solutions under the hypothesis $c_i > 0$, $d_i > 0$, $E_i \geq 0$, $a_i \geq 0$ and $b_i \geq 0$, $i = 1,2$. Under the additional hypothesis $c_1 = c_2$, we prove the global existence of solutions by energy estimates.

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